"Binary Quadratic Forms as Equal Sums of Like Powers"

Titus Piezas III

Abstract: A summary of some results for binary quadratic forms as equal sums of like powers will be given, in particular several *sum-product* identities for higher powers. A conjecture will also be raised, including its context in terms of the general *Euler's Extended Conjecture*.

Contents:

I. Introduction

In "*Ramanujan and Fifth Power Identities*" [1] we gave some identities that depended on solving the general equation,

$$ax^2+bxy+cy^2 = dz^2 \qquad \text{(eq.1)}$$

To recall,

$$F(x,y) = ax^2+bxy+cy^2$$

is a *binary quadratic form* associated with a 2x2 matrix. In particular, the forms a^2+b^2 and a^2+ab+b^2, connected to the Gaussian and Eisenstein integers, were much discussed in [1].

For a particular $\{a,b,c,d\}$, it may be desired to find a parametric solution to eq.1. Given one solution, while there are known techniques to generate an infinite more, in this paper we will use a "template", a *sum-product* identity, such that the initial solution is enough to provide a parametrization. Surprisingly, various forms of these sum-products as binary quadratic forms exist for degrees k = 2,3,4,5 and are *multi-grade* for the higher degrees. They are then very useful in generating parametrizations for equal sums of like powers. Whether there are similar ones for higher powers remains to be seen.

II. 2nd Powers: The Form $ax^2+bxy+cy^2 = dz^2$

For second powers, one sum-product identity for {*a,b,c,d*} is given by,

$$ax^2+bxy+cy^2-dz^2 = (am^2+bmn+cn^2-dp^2)(au^2+buv+cv^2)^2$$

where,

$$x = (am+bn)u^2 + 2cnuv - cmv^2$$
$$y = -anu^2 + 2amuv + (bm+cn)v^2$$
$$z = p(au^2 + buv + cv^2)$$

for arbitrary variables *u,v*. The problem of finding a parametrization for *x,y,z* is then reduced to finding one initial solution *m,n,p*. Notice that the sum *z* will always be a multiple of the initial sum *p*. This is a peculiarity of this particular formula since it is just a special case of an even more general identity which has *four* free parameters. Another version can be given as,

$$ax^2+bxy+cy^2-dz^2 = (am^2+bmn+cn^2-dp^2)(-u^2+(a+b+c)dv^2)^2$$

where,

$$x = mu^2 + 2dpuv + (am-cm+bn+2cn)dv^2$$
$$y = nu^2 + 2dpuv + (2am+bm-an+cn)dv^2$$
$$z = pu^2 + (2am+bm+bn+2cn)uv + (a+b+c)dpv^2$$

Examples. Given $u^2+uv+v^2 = 7w^2$, one solution is (*m,n,p*) = (2,1,1). Using these values on the two identities, we find,

$$x = 3u^2+2uv-2v^2, \quad y = -u^2+4uv+3v^2, \quad z = u^2+uv+v^2$$

$$x = 2u^2+14uv+21v^2, \quad y = u^2+14uv+42v^2, \quad z = u^2+9uv+21v^2.$$

III. 3rd, 4th, 5th Powers: Sum-Product Identities

The most general form for equal sums of like powers is

$$a_1^k + a_2^k + \ldots a_m^k = b_1^k + b_2^k + \ldots b_n^k$$

denoted as *k.m.n*. There are some beautiful parametrizations for the special case of the *k.1.k*, or *k k*th powers equal to a *k*th power given by,

Vieta, 1591:

$$a^3(a^3-2b^3)^3 + b^3(2a^3-b^3)^3 + b^3(a^3+b^3)^3 = a^3(a^3+b^3)^3$$

Fauquembergue, 1898:

$$(4x^4-y^4)^4 + (4x^3y)^4 + (4x^3y)^4 + (2xy^3)^4 + (2xy^3)^4 = (4x^4+y^4)^4$$

Sastry, 1934:

$$(u^5+25v^5)^5 + (u^5-25v^5)^5 + (10u^3v^2)^5 + (50uv^4)^5 + (-u^5+75v^5)^5 = (u^5+75v^5)^5$$

Presented in this manner, it is certainly suggestive what the identity for the next power would look like, though nothing of comparable simplicity is known for sixth powers and higher for a *minimum* number of terms. (It gets easier the more terms there are.) For the following identities, as was mentioned in the Introduction, we will not be using higher order polynomials for the addends. Two kinds will be given, a plain sum in linear forms and a sum-product in quadratic forms, both of which using the same expressions. These are:

A. 3rd Powers, k.4.4:

$$(ax+v_1y)^k + (bx-v_1y)^k + (cx-v_2y)^k + (dx+v_2y)^k = (ax-v_1y)^k + (bx+v_1y)^k + (cx+v_2y)^k + (dx-v_2y)^k$$

$$(ax^2-v_1xy+bwy^2)^k + (bx^2+v_1xy+awy^2)^k + (cx^2+v_2xy+dwy^2)^k + (dx^2-v_2xy+cwy^2)^k = (a^k+b^k+c^k+d^k)(x^2+wy^2)^k$$

for k = 1,3, where $\{v_1, v_2\} = \{c^2-d^2, a^2-b^2\}$, and $w = (a+b)(c+d)$, for six arbitrary variables a,b,c,d,x,y.

B. 4th Powers, k.3.3:

$$(ax+v_1y)^k + (bx-v_2y)^k + (cx-v_3y)^k = (ax-v_1y)^k + (bx+v_2y)^k + (cx+v_3y)^k$$

$$(ax^2+2v_1xy-3ay^2)^k + (bx^2-2v_2xy-3by^2)^k + (cx^2-2v_3xy-3cy^2)^k = (a^k+b^k+c^k)(x^2+3y^2)^k$$

for k = 2,4, where$\{v_1, v_2, v_3, c\} = \{a+2b, 2a+b, a-b, a+b\}$, and for four arbitrary variables a,b,x,y.

C. 5th Powers, k.6.6:

$$(a_1x+v_1y)^k + (a_2x-v_2y)^k + (a_3x+v_3y)^k + (a_4x-v_3y)^k + (a_5x+v_2y)^k + (a_6x-v_1y)^k = (a_1x-v_1y)^k + (a_2x+v_2y)^k + (a_3x-v_3y)^k + (a_4x+v_3y)^k + (a_5x-v_2y)^k + (a_6x+v_1y)^k$$

$$(a_1x^2+2v_1xy+3a_6y^2)^k + (a_2x^2-2v_2xy+3a_5y^2)^k + (a_3x^2+2v_3xy+3a_4y^2)^k + (a_4x^2-2v_3xy+3a_3y^2)^k + (a_5x^2+2v_2xy+3a_2y^2)^k + (a_6x^2-2v_1xy+3a_1y^2)^k = (a_1^k+ a_2^k+a_3^k+a_4^k+a_5^k+a_6^k)(x^2+3y^2)^k$$

for k = 1,2,3,4,5, where $\{a_1, a_2, a_3, a_4, a_5, a_6\} = \{a+c, b+c, -a-b+c, a+b+c, -b+c, -a+c\}$, and $\{v_1, v_2, v_3\} = \{a+2b, 2a+b, a-b\}$ for five arbitrary variables a,b,c,x,y. (Note that the v_i are precisely the same as the ones for fourth powers.)

Example. Using initial solutions,

$$3^3 + 4^3 + 5^3 = 6^3$$

$$2^4 + 2^4 + 3^4 + 4^4 + 4^4 = 5^4$$

$$(-1)^5 + (-1)^5 + 1^5 + 1^5 + 3^5 + 3^5 = 2(3)^5$$

(with the third one using $\{a,b,c\} = \{-2,2,1\}$), we then get the parametrizations,

$$(3x^2+5xy-5y^2)^3 + (4x^2-4xy+6y^2)^3 + (5x^2-5xy-3y^2)^3 = (6x^2-4xy+4y^2)^3$$

$$(2x^2+12xy-6y^2)^4 + (2x^2-12xy-6y^2)^4 + (3x^2+9y^2)^4 + (4x^2-12y^2)^4 + (4x^2+12y^2)^4 = (5x^2+15y^2)^4$$

$$(-x^2+4xy+9y^2)^k + (-x^2-4xy+9y^2)^k + (x^2+8xy+3y^2)^k + (x^2-8xy+3y^2)^k + (3x^2+4xy-3y^2)^k + (3x^2-4xy-3y^2)^k = 2(3x^2+9y^2)^k, \quad \text{for } k = 1,3,5$$

One important difference that can be pointed out is that for higher powers k = 4,5, *the solutions most be of a certain form* (such as $a+b = c$) while for k = 2,3, there is no side condition that must be followed. For k = 3, since the complete parametrization of *cubic quadruples* $a^3 + b^3 + c^3 + d^3 = 0$ is known (due to Euler), then the identity provides a quadratic form parametrization for any $\{a,b,c,d\}$ though like k = 2, the formula given here is just a special case of something even more general. It obviously applies to cubic n-tuples. Going to higher powers, if there is a sum-product for k = 6, then there must be conditions on the solutions as well, though what those would be are still unknown. For more details, particularly the derivation, see the following papers by this author: "*Ramanujan and the Cubic Equation* $3^3 + 4^3 + 5^3 = 6^3$", "*Ramanujan and the Quartic Equation* $2^4+2^4+3^4+4^4+4^4=5^4$", and "*Ramanujan and Fifth Power Identities*" with the first one spelling out the procedure as the latter ones were essentially by analogy. For a list of small primitive solutions to (4,1,5), see "*On the Clustering of Sums of Quartic Sextuples*" all at http://www.geocities.com/titus_piezas/ramanujan.html.

IV. 6th, 7th Powers in Quadratic Forms

In this section we'll give a collection of some elegant parametrizations for higher powers in terms of binary quadratic forms:

A. 5th Powers

$$(a+c)^k + (b+c)^k + (a+b+c)^k + (-a-b+c)^k + (-b+c)^k + (-a+c)^k = 2(3c)^k$$

where $a^2+ab+b^2 = 4c^2$, and,

$$(a+c)^k + (b+c)^k + (a+b+c)^k + (-a-b+c)^k + (-b+c)^k + (-a+c)^k + c^k = (7c)^k$$

where $a^2+ab+b^2 = 28c^2$, both for $k = 1,3,5$.

B. 6th Powers (Chernick, 1937)

$$(u-7w)^k + (u-2v+w)^k + (3u+w)^k + (3u+2v+w)^k = (u+7w)^k + (u-2v-w)^k + (3u-w)^k + (3u+2v-w)^k$$

for $k = 2,4,6$ where $u^2+uv+v^2 = 7w^2$.

C. 7th Powers (Sinha, 1966)

$$(-a+c)^k + (-a-c)^k + (a+2b+c)^k + (a+2b-c)^k + (a-b)^k + (3a+b)^k + (-a-13b)^k + (2a+9b)^k + (-2a+3b)^k = (3(a+b))^k$$

for $k = 1,3,5,7$ where $-5a^2+10ab+121b^2 = c^2$.

D. 8th Powers (Sinha, 1966)

$$(a-r)^k + (a+r)^k + (3b-t)^k + (3b+t)^k + (4a)^k = (b-t)^k + (b+t)^k + (3a-r)^k + (3a+r)^k + (4b)^k$$

for $k = 1,2,4,6,8$ where $a^2+12b^2 = r^2$, and $12a^2+b^2 = t^2$.

Some points: First, the one for seventh powers was modified by this author to present it in a more aesthetic manner, though the one by Sinha was more illustrative of how it was derived. Second, the one for eight powers is no longer in quadratic forms as there are *two* quadratic conditions but it was included here because it is a very nice identity. For the rest, obviously one only need to find an initial solution and using either of the two identities discussed in the section on 2nd powers, a parametrization can then be found.

V. Conclusion

Among the various cases of equal sums of like powers, the *balanced* case *k.m.m* is the easiest to work with since we can exploit certain symmetries of having an equal number of terms on both sides. However, the less the number of terms, the harder it is to find solutions. Known results for binary quadratic forms of the case *k.m.m* are as follows:

{2.1.1}, {3.2.2}, {4.3.3}, {5.4.4}, {6.4.4}, {7.5.5}

and which lead us to the questions,

1. *Given the balanced equation k.m.m where the terms are binary quadratic forms, what is the* ***minimum*** *m for each k > 3?*
2. *For even k = 2h, is it {2h, h+1, h+1}?*
3. *For odd k = 2h+1, is it {2h+1, h+2, h+2}?*

This is then a quadratic forms version, limited only to the balanced case, of *Euler's Extended Conjecture* (EEC), a generalization of Fermat's Last Theorem and formalized by Ekl (1998). This conjecture states that for *k* > *m+n*, the equation *k.m.n* has no non-trivial solution in the integers. For its balanced case, minimum *m* is then conjectured to be,

Even k = 2h: {2h, h, h}
Odd k = 2h+1: {2h+1, h+1, h+1}

Compare results for quadratic forms. A stronger version of EEC was formalized by Meyrignac (1999) as the *Lander-Parkin-Selfridge Conjecture* and which states that given a non-trivial integral solution to *k.m.n*, then for every *k* > 3, *m+n* >= *k*. Not much is known about these conjectures in the integers and even less as quadratic forms. As the former, examples for 7.4.4 are known, some of which by Choudhry are multi-grade, but none yet for 8.4.4 (though a few 9.5.5 are known). As quadratic forms, 7.5.5 is known but not yet, if any, for 8.5.5 though Sinha's octic parametrization came tantalizingly close. Considering that 7.5.5 was found way back in 1966, it seems a disgrace no other identity for 8.5.5, or higher, with terms as polynomials of small degree are known, with all the information and computing power at our disposal.

Perhaps the next few decades for equal sums of like powers will be different.

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titus_piezasIII@yahoo.com (Pls. remove "III")
www.geocities.com/titus_piezas/

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