"A New Way To Solve Cubics Using A Linear Fractional Transformation"

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ABSTRACT: We provide a new method to solve the general cubic equation by using a linear fractional transformation. This transformation, sometimes referred to as a Möbius transformation, can transform the general cubic into the binomial form, though in a manner different from the traditional Tschirnhausen transformation that can also transform the cubic into the binomial form. The resulting binomial is then simply solved by the extraction of a cube root.

Mathematics Subject Classification. Primary: 12E12.

Babylon, c. 1800 BC

The scribe was finishing his work on the tablet. He put down his tools for a moment, lightly brushed away stray clay bits from some of the still-wet, wedge-shaped notches with his finger, then held the tablet at arm's length to admire his handiwork.

It had four columns with fifteen rows. The second and third columns were numbers that belong to what millennia later would be called "Pythagorean triples".

The first and thirteenth rows were his favorite, containing symbols equivalent to "119, 169" and "161, 289". So, 169^2-$119^2 = 120^2$ and 289^2-$161^2 = 240^2$.

But it was also equivalent to 13^4-$119^2 = 120^2$ and 17^4-$161^2 = 240^2$ so the first number was *doubly square.* He wanted to find more relations like these but these two were the only ones he found so far.

Were there infinite numbers like these, he wondered? He shrugged, thought maybe no wise master of numbers could ever answer a question like that, and went back to finishing the tablet…

Almost four thousand years later, the tablet will find its way across a vast ocean and onto a New World. It will be kept in a place called "Columbia University" and be called the Plimpton 322…

I. Introduction

After the Babylonians discovered the quadratic formula c. 2000 BC, or at least a version of it, it would not be surprising if pioneering mathematicians would wonder about the cubic case, especially if it arose in a geometric context. In fact, the Babylonians themselves did consider certain cubic equations which arose in trying to solve problems involving volume.

The ancient Greeks also worked on particular cubic equations. One of the early mathematicians who considered the cubic was Diophantus of Alexandria (c. 200-284 AD), better known for his work on what are known as Diophantine equations. Centuries later, we have the Persian poet-mathematician Omar Khayyam (1048-1131) who managed to find the solution to the cubic $x^3 + 200x = 20x^2 + 200$ and similar equations using conic sections.

Leonardo Fibonacci (1170-1250), of the Fibonacci numbers, also did work on cubics, solving $x^3 + 2x^2 + 10x = 20$ and other equations of similar form. However, not all mathematicians of the time believed that the *general* cubic was solvable. The Italian mathematician Luca Paciola (1445-1509) thought that it was impossible to solve because of the very good reason that he could not do it.

One person who could solve the cubic was Scipione del Ferro (1465-1526), though his method worked on the *depressed or reduced* case, a cubic with no x^2 term. As could Niccolo "Tartaglia" Fontana (1499-1557), who independently came up with a solution. While initially thought as still short of the general case, the solution of the reduced cubic was in fact all that was needed, as was realized by Girolamo Cardano (1501-1576). He published the definitive cubic formula in a popular book to the dismay, to use a mild word, of Tartaglia, upon whose work Cardano had built on.

II. The Traditional Method

Before we go to our new method to solve the cubic, perhaps we can give a version of Cardano's formula, slightly modified with insights from later mathematicians, especially Joseph-Louis Lagrange (1736-1813). What Lagrange realized was that to solve equations of prime degree *n* with rational coefficients, one has to solve a *resolvent* equation of degree *n-1* also with rational coefficients, which are now called *Lagrange resolvents*. Thus, to solve the cubic, one has to first solve a quadratic.

Given the general cubic,

$$x^3 + ax^2 + bx + c = 0$$

its resolvent equation is given by,

$$z^2 + (2a^3 - 9ab + 27c)z + (a^2 - 3b)^3 = 0$$

such that the solution to the cubic is,

$$x = \frac{-a + z_1^{1/3} + z_2^{1/3}}{3}$$

where the z_i are the two roots of the resolvent.

One can note two things. First, the solution is analogous to the quadratic formula. And while Galois theory has established that formulas using a finite number of arithmetic operations and root extractions are impossible for *general* equations of degree greater than four, there are *particular* equations solvable as such. So the solution to the *solvable quintic* should be similar in form, namely,

$$x = \frac{-a + z_1^{1/5} + z_2^{1/5} + z_3^{1/5} + z_4^{1/5}}{5}$$

where the z_i are the roots of its quartic Lagrange resolvent. And likewise for the solvable septic,

$$x = \frac{-a + z_1^{1/7} + z_2^{1/7} + z_3^{1/7} + z_4^{1/7} + z_5^{1/7} + z_6^{1/7}}{7}$$

where the z_i are the roots of its sextic Lagrange resolvent (see examples for the quintic and septic in *"An Easy Way To Solve Solvable Quintics Using Two Sextics"* by the same author.) The same goes for the solvable 11th-degree, 13th-degree, and other solvable equations of prime degree.

Second, the constant term of the cubic's Langrange resolvent is a *cube power*. It should not be hard to guess that the constant term of the quintic's Lagrange resolvent is a fifth power, and so on for the other *odd* prime degrees. One can only marvel at the aesthetics of such mathematical consistency, an insight which we can write down in a few lines, but took many mathematicians many centuries to discover.

The problem, of course, is to find these Lagrange resolvents, and one can show that these depend on an *auxiliary* resolvent equation of *(n-2)!* degree, which complicates matters for prime $n > 3$, though we shall reserve more discussion on this for another time.

For an example to the cubic formula, let,

$$x^3 - 5x^2 + x - 7 = 0$$

Resolvent is,

$$z^2 - 394z + 22^3 = 0$$

with roots,

$$z_i = 197 \pm 3\sqrt{3129}$$

thus,

$$x = \frac{5 + (197 + 3\sqrt{3129})^{1/3} + (197 - 3\sqrt{3129})^{1/3}}{3} = 5.07475...$$

III. The New Method

We mentioned earlier the *depressed or reduced cubic*, or a cubic with no x^2 term. The general cubic can easily be transformed into that form using a simple transformation known as a *translation*.

Given,

$$x^3 + ax^2 + bx + c = 0$$

we do the substitution $x = y+p$, for some indeterminate p. Expanding and collecting the new variable y, we have,

$$y^3 + (3p + a)y^2 + (3p^2 + 2ap + b)y + (p^3 + ap^2 + bp + c) = 0$$

Translating an equation means to find its *shifted* version with no rotation or distortion. If we eliminate its y^2 term, then it is equivalent to shifting the graph of the equation such that the sum of its roots is $x_1 + x_2 + x_3 = 0$. However, if so desired, we can also eliminate its y term. By equating any of the coefficients other than the constant term to zero, one can eliminate either the y^2 or y term by solving for p using an equation less than a cubic. The method can obviously be applied to any *nth* degree equation to eliminate any of its coefficients (other than the constant term) by solving an equation of degree less than n.

If we want to *simultaneously* eliminate both the y^2 or y term, then we set,

$$3p + a = 0 \qquad\qquad\qquad (eq.1)$$
$$3p^2 + 2ap + b = 0 \qquad\qquad (eq.2)$$

So, $p = $ *-a/3*. Substituting this value into (eq.2), we have,

$$a^2 - 3b = 0$$

If the coefficients a and b of the cubic satisfy the above equation, then by doing a translation to eliminate the y^2 term, it also automatically eliminates the y term. Perhaps one can then use a *prior* translation to set $a^2 - 3b = 0$. Let $x = z+q$ where q is an indeterminate variable. Collecting z, we have,

$$z^3 + (3q + a)z^2 + (3q^2 + 2aq + b)z + (q^3 + aq^2 + bq + c) = 0$$

We wish to set,

$$(3q + a)^2 - 3(3q^2 + 2aq + b) = 0$$

If we try to solve for q,

$$9q^2 + 6aq + a^2 - (9q^2 + 6aq + 3b) = 0$$

it resolves to,

$$a^2 - 3b = 0$$

the variable q disappears and we are left with a condition which was what we wanted to accomplish in the first place. Thus we cannot use a simple translation to simultaneously eliminate two terms of the *general* cubic.

There is a transformation, known as a *Tschirnhausen transformation*, (after Count Ehrenfried Walter von Tschirnhaus, 1651-1708), that can eliminate two terms of the general

cubic and higher degrees. In fact, it can simultaneously eliminate in radicals as much as the three terms x^{n-1}, x^{n-2}, x^{n-3} for *n* equal to four and higher. We shall not discuss the basics of this transformation here, though the interested reader is referred to the paper *"Solving Solvable Quintics Using One Fifth Root Extraction"* by the same author for more details.

What we would like to discuss though is the possibility that there might be a non-Tschirnhausen transformation that can reduce the cubic to binomial form using only the elementary transformations of translation, dilation, inversions, etc. A translation is an example of an *affine transformation.* In general, affine transformations are compositions of translations, rotations, dilations, and shears. They preserve 1) collinearity (all points initially lying on a line still lie on a line after transformation) and 2) ratios of distances. They do not necessarily preserve angles.

However, there are certain transformations that generally preserve angles. Such transformations are called *conformal transformations* or *conformal mappings*, an example of which is the *linear fractional transformation (LFT)*:

$$x = \frac{az + b}{cz + d}$$

where, *a, b, c, d* are complex numbers and,

$$ad - bc \neq 0$$

An LFT is a composition of translations, rotations, and *inversions* (additive or multiplicative) and sends circles and lines to circles or lines. The above is also known as a *Möbius transformation*, after the mathematician August Ferdinand Möbius (1790-1868) better known for the Möbius strip.

We have italicized the operation of inversion to highlight its role. If we also *flip over* the cubic equation, so to speak, or find the multiplicative inverse of the translation $x = \frac{1}{z + q}$,

perhaps we can set $a^2 - 3b$. After it has been "conditioned", we still need another translation to eliminate z^2 and *z*, so the complete transformation is $x = \frac{1}{z + q} + p$, which is equivalent to the

LFT $x = \frac{pz + (pq + 1)}{z + q}$. The two unknowns *p* and *q* then give us two degrees of freedom.

To do the transformation, we substitute $x = \frac{1}{z + q} + p$ into the general cubic, expand and

collect the new variable *z*, and we get,

$$z^3 + \left(\frac{3r_1 q + r_2}{r_1} \right) z^2 + \left(\frac{3r_1 q^2 + 2r_2 q + r_3}{r_1} \right) z + \left(\frac{r_1 q^3 + r_2 q^2 + r_3 q + 1}{r_1} \right) = 0$$

where the r_i are,

$$r_1 = p^3 + ap^2 + bp + c$$

$$r_2 = 3p^2 + 2ap + b$$

$$r_3 = 3p + a$$

The first step, we set,

$$\left(\frac{3r_1q + r_2}{r_1}\right)^2 - 3\left(\frac{3r_1q^2 + 2r_2q + r_3}{r_1}\right) = 0$$

which resolves to,

$$r_2{}^2 - 3r_1r_3 = 0 \qquad\qquad \text{(eq.3)}$$

substituting the r_i into (eq.3), we get,

$$(a^2 - 3b)p^2 + (ab - 9c)p + b^2 - 3ac = 0$$

and we have the equation for the unknown p in terms of the cubic's coefficients!

Solving for p and getting the negative case, we have,

$$p = \frac{-ab + 9c - \sqrt{(ab - 9c)^2 - 4(a^2 - 3b)(b^2 - 3ac)}}{2(a^2 - 3b)}$$

And we have our first unknown. To get the second, we eliminate the coefficient z^2 by letting,

$$3r_1q + r_2 = 0$$

Solving for q,

$$q = -\frac{r_2}{3r_1}$$

Substituting the r_i and the value of p and simplifying, we get,

$$q = \frac{a^2 - 3b}{\sqrt{(ab - 9c)^2 - 4(a^2 - 3b)(b^2 - 3ac)}}$$

and we now have our two unknowns! We know that by eliminating z^2, since we have "conditioned" the equation, it will also automatically eliminate the z term, and we have attained our objective of reducing the general cubic into binomial form using a non-Tschirnhausen transformation.

For non-zero a, we will not write down in symbolic form the constant term of this cubic binomial as it too unwieldy. What we only need to know is that given the linear fractional

transformation $x = \dfrac{1}{z+q} + p$, with p and q as defined above, then the cubic will be transformed into the binomial form. For $a = 0$, we can give the explicit expression for z as it is simpler. Given,

$$x^3 + bx + c = 0$$

then,

$$x = \frac{1}{z+q} + p$$

where,

$$p = \frac{-9c + \sqrt{3d}}{6b}, \qquad q = \frac{-3b}{\sqrt{3d}}, \qquad z = -\left(\frac{27cd + 3(2b^3 + 27c^2)\sqrt{3d}}{2d^2}\right)^{1/3}$$

with the *discriminant* $d = 4b^3 + 27c^2$.

IV. Examples

Example 1.

We can use the same cubic cited earlier to serve as comparison. Given,

$$x^3 - 5x^2 + x - 7 = 0$$

Using the identities for p and q in terms of the cubic's coefficients, we have,

$$p = \frac{-29 - \sqrt{3129}}{22}, \qquad q = \frac{11}{\sqrt{3129}}$$

let $x = \dfrac{1}{z+q} + p$ and the cubic transforms into the binomial in the variable z as,

$$z^3 - \frac{1849239 - 33485\sqrt{3129}}{6258^2} = 0$$

so a solution to the cubic is given by,

$$x = \frac{1}{\left(\dfrac{1849239 - 33485\sqrt{3129}}{6258^2}\right)^{1/3} + \dfrac{11}{\sqrt{3129}}} + \frac{-29 - \sqrt{3129}}{22} = 5.07475...$$

Example 2.

For the second example, we will use a reduced or depressed cubic. Given,

$$x^3 + 2x + 4 = 0$$

since we have the explicit expressions for *p, q, z*, then we find one solution as,

$$x = -3 + \sqrt{\frac{29}{3} - \frac{58}{\sqrt{87} + \sqrt[3]{87(261 + 28\sqrt{87})}}} = -1.179509\ldots$$

where the other two solutions for both examples can be given by using the complex roots of the cube root of unity.

V. Conclusion

In this paper, we have established a new method to find exact solutions to the cubic equation. However, some points should be made. Since the variables *p* and *q* have denominators, then there is the possibility of division by zero. Thus, while the method can be applied *to the general cubic with unconditional symbolic coefficients*, it cannot be applied to a cubic with coefficients that have certain conditions.

The variable *p* has the denominator $a^2 - 3b$. If the cubic to be solved already has coefficients such that $a^2 - 3b = 0$, then we will have division by zero. However, since the essence of our method consists of two steps, a) to set the cubic into something with precisely that condition and b) a translation that simultaneously eliminates the z^2 and *z* term, then one can just skip the first step.

The variable *q* has the denominator $\sqrt{(ab - 9c)^2 - 4(a^2 - 3b)(b^2 - 3ac)}$. The expression $(ab - 9c)^2 - 4(a^2 - 3b)(b^2 - 3ac)$ is in fact the *discriminant* of the general cubic. This is zero if and only if the cubic has repeated roots. If the coefficients are rational, then the cubic will factor into rational factors.

The role of the new method then is not intended to be a mere root-finder; Tartaglia‑Cardano's formula is more suited for that. Rather, it primarily serves to elucidate a connection between cubic equations and conformal mappings, particularly linear fractional transformations (LFTs). Such transformations that transform objects in space to similar objects have connections to fractals and iterated functions. In fact, certain iterates of LFTs are just the convergents to simple continued fractions.

It should be interesting to investigate the associated continued fractions of the iterates of LFTs that reduce cubics to binomial form, though that would be already outside the scope of this paper. If this paper motivates such investigation, then it would have served it purpose.

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Author's Note:

For more on the Plimpton 322 tablet, see
http://aleph0.clarku.edu/~djoyce/mathhist/plimpnote.html

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