The Equation $q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)q_5^3$ in Quadratic forms q_i

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Contents: diophantine equations, equal sums of like powers, cubic quadruples.

This is a very short paper that is essentially an epilogue to "*Ramanujan and the Cubic Equation* $3^3 + 4^3 + 5^3 = 6^3$" [1]. In that paper, we found an algebraic identity such that given a solution to the cubic quadruple $a^3 + b^3 + c^3 + d^3 = 0$ we can automatically find one in terms of quadratic forms. However, we mentioned it seems there was an even more general identity that contains the one we gave. In this continuation, we finally find this general form.

Our method consists of solving the equation,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = 0 \qquad \text{(eq.1)}$$

where the q_i are quadratic forms. Explicitly,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = 0$$

where the v_i are unknowns. By expanding and collecting terms in powers of x and y, we get the system of equations, call it S_1,

$$p_0 = (a^3 + b^3 + c^3 + d^3)\, x^6$$
$$p_1 = (a^2v_1 + b^2v_3 + c^2v_5 + d^2v_7)\, x^5y$$
$$p_2 = (av_1^2 + a^2v_2 + bv_3^2 + b^2v_4 + cv_5^2 + c^2v_6 + dv_7^2 + d^2v_8)\, x^4y^2$$
$$p_3 = (v_1^3 + 6av_1v_2 + v_3^3 + 6bv_3v_4 + v_5^3 + 6cv_5v_6 + v_7^3 + 6dv_7v_8)\, x^3y^3$$
$$p_4 = (v_1^2v_2 + av_2^2 + v_3^2v_4 + bv_4^2 + v_5^2v_6 + cv_6^2 + v_7^2v_8 + dv_8^2)\, x^2y^4$$
$$p_5 = (v_1v_2^2 + v_3v_4^2 + v_5v_6^2 + v_7v_8^2)\, xy^5$$
$$p_6 = (v_2^3 + v_4^3 + v_6^3 + v_8^3)\, y^6$$

The details can be found in [1] but by equating all $p_i = 0$ and solving for the v_i, we get what we called Id.2:

Identity 2 (Id.2)

If $a^3 + b^3 + c^3 + d^3 = 0$, then,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = 0$$

with the solution, using the simplifying substitution $a = p+s$, $b = p-s$, $c = q-r$, $d = q+r$,

v_1, v_3 (free variables)

$$v_5\,(6pqr) = (p^3+q^3+3q^2r+3p^2s)v_1 + (p^3+q^3+3q^2r-3p^2s)v_3$$

$$v_7\,(6pqr) = -(p^3+q^3-3q^2r+3p^2s)v_1 - (p^3+q^3-3q^2r-3p^2s)v_3$$

$$v_1^2-4av_2 = f_1, \quad v_3^2-4bv_4 = f_1, \quad v_5^2-4cv_6 = f_2, \quad v_7^2-4dv_8 = f_2$$

and discriminants f_i,

$$f_1 = -(4p^3+q^3)(pv_1-sv_1-pv_3-sv_3)^2/(12p^2qr^2)$$

$$f_2 = -(p^3+4q^3)(pv_1-sv_1-pv_3-sv_3)^2/(12pq^2r^2)$$

One can easily solve for the v_i with even subscripts as they are only linear (and since the v_i with odd subscripts are already given). The new variables (p,q,r,s) can be expressed in terms of the originals as,

$$p = (a+b)/2, \quad s = (a-b)/2, \quad q = (c+d)/2, \quad r = (-c+d)/2$$

If we substitute the formulas for the v_i into eq.1, what we get is the factorization,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)\, P(x,y)$$

We pointed out that P(x,y) is a complicated polynomial, *not a perfect cube*, that is irrelevant if $a^3 + b^3 + c^3 + d^3 = 0$ since the right hand side of the equation vanishes anyway. It was not a cube using these particular versions for the expressions for the v_i but with a small substitution, *it turns out we can make this into a cube!*

It was simple, really. Using the simplifying substitutions, then $a^3 + b^3 + c^3 + d^3 = 0$ becomes,

$$p^3 + q^3 + 3qr^2 + 3ps^2 = 0$$

or $p^3 + q^3 = -(3qr^2 + 3ps^2)$. Since v_5 is given by,

$$v_5\,(6pqr) = (p^3+q^3+3q^2r+3p^2s)v_1 + (p^3+q^3+3q^2r-3p^2s)v_3$$

then this is equivalent to,

$$v_5\,(2pqr) = (-qr^2-ps^2+q^2r+p^2s)v_1 + (-qr^2-ps^2+q^2r-p^2s)v_3$$

Same for v_7. (I actually did this once before. But I neglected to do the same for the discriminants f_i!) Since f_1 is,

$$f_1 = -(4p^3+q^3)(pv_1-sv_1-pv_3-sv_3)^2/(12p^2qr^2)$$

then $f_1 = -(p^3-qr^2-ps^2)(pv_1-sv_1-pv_3-sv_3)^2/(4p^2qr^2)$ and similarly for f_2. Substituting these new versions for the v_i, we get the fifth identity *which covers all four* given in [1]:

Identity 5 (Id.5)

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)q_5^3 \qquad \text{(eq.2)}$$

or, explicitly,

$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = (a^3 + b^3 + c^3 + d^3)(x^2 + v_9xy + v_{10}y^2)^3$

where, with the simplifying substitution a = p+s, b = p-s, c = q-r, d = q+r,

v_1, v_3 (free variables)

$v_5\,(2pqr) = (qr(q-r)+ps(p-s))v_1 + (qr(q-r)-ps(p+s))v_3$

$v_7\,(2pqr) = (qr(q+r)-ps(p-s))v_1 + (qr(q+r)+ps(p+s))v_3$

$v_9 = (v_1+v_3)/(2p)$

$v_1^2-4av_2 = f_1$, $v_3^2-4bv_4 = f_1$, $v_5^2-4cv_6 = f_2$, $v_7^2-4dv_8 = f_2$, $v_9^2-4v_{10} = f_3$

$f_1 = f_3(p^3-qr^2-ps^2)/p$, $f_2 = f_3(q^3-qr^2-ps^2)/q$, $f_3 = -(pv_1-sv_1-pv_3-sv_3)^2/(4pqr^2)$

and,

p = (a+b)/2, s = (a-b)/2, q = (c+d)/2, r = (-c+d)/2

This is a much more satisfying version as this can be applicable to cubic n-tuples *without conditions on (a,b,c,d) and dependence on opposite parity of middle terms*, other than the old assumption that pairs of quadratic forms on the left hand side, (q_1, q_2) and (q_3, q_4), share the same discriminants. And it also answers one question in the conclusion of [1]: whether S_2, or the system of equations we get by expanding (eq.2) plus two more equating the discriminants, always has a linear solution. So the answer indeed is yes.

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http://www.geocities.com/titus_piezas/ramanujan.html → Click here for an index
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Reference:

1. Piezas, T., "Ramanujan and The Cubic Equation $3^3 + 4^3 + 5^3 = 6^3$", http://www.geocities.com/titus_piezas/ramanujan_page9.html