"Solving Solvable Quintics Using One Fifth Root Extraction"

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ABSTRACT: We prove that all irreducible but solvable equations of degree n can be transformed *in radicals* into the binomial form $y^n+c=0$ using a Tschirnhausen transformation of degree $n-1$. The resulting equation is then solvable by a single *nth* root extraction. In particular, we illustrate the method using the solvable quintic.

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The old man stood by the window of his study, looking out at the grounds of his estate. He barely noticed the light snow falling outside, as he was lost in thought. Things hadn't turned out that well in recent years and now he was deeply in debt.

He was of noble blood, a count in fact, but he was also a scientist, a philosopher, as well as being a mathematician. However, it seems he was not completely successful in any of his roles.

As a scientist, his experiments in porcelain were promising. Most of the porcelain came from China, and an efficient way to locally make "white gold", as porcelain was known, would ensure the family fortune and plans were made for a factory. However, the war with Sweden, among other things, had disrupted his schedule.

As a philosopher, he was being eclipsed by the Dutch philosopher Benedictus Spinoza with whom he had engaged in a correspondence. And as a mathematician, he had hoped to make his mark by finding a general method to solve equations of any degree, discussed in a paper he wrote back in 1683. However, the philosopher-mathematician Gottfried Liebniz had pointed out certain difficulties with his method.

He gave a deep sigh, turned around and walked towards his desk. He picked up a delicate porcelain vase one of his craftsmen had made. It was decorated with paintings of grape vines.

Perhaps one day, he mused, one of my ideas will also bear fruit.

Almost three hundred years later, in a certain tropical country south-east of China…

I. Introduction

In this paper, we will answer two related questions. First, can we turn any irreducible solvable equation Q(x) into the binomial form $y^n + c_0 = 0$ in radicals using a Tschirnhausen transformation? For example, can we turn the solvable quintic,

$$x^5 - 2x^4 + 3x^3 - 3x^2 + x + 1 = 0$$

into the binomial form,

$$y^n + c_0 = 0$$

for some constant c_0? Second, the generalization of the first, can we turn a solvable equation Q(x) with no repeated roots into *any* solvable form P(y) of the same degree in radicals using a Tschirnhausen transformation? For example, using the same solvable quintic given above:

$$x^5 - 2x^4 + 3x^3 - 3x^2 + x + 1 = 0$$

which has discriminant $D = 5^{15}103^2$, can we turn it into *any* arbitrary solvable quintic, say,

$$y^5 - 2y^4 + 2y^3 - y^2 + 1 = 0$$

with discriminant $D = 5^{15}47^2$ in radicals? Or equivalently, can we express the roots y_i of the latter in terms of the roots x_i of the former?

The answer is *yes*, for both questions and the proof, in fact is quite simple.

II. The Tschirnhausen Transformation

One of the crucial insights into solving the general cubic way back in the 16th century was that it was possible to *depress* or reduce the cubic such that it had no x^2 term. Niccolo "Tartaglia" Fontana (1499-1557) could solve cubics of the form,

$$x^3 + px = q$$

and while initially thought as not being a general solution, in fact, it was all that was needed, as was realized by Girolamo Cardano (1501-1576). Given the general cubic,

$$x^3 + ax^2 + bx + c = 0$$

we do the substitution $x = y+r$, for some indeterminate r. Expanding and collecting, we have,

$$y^3 + (3r + a)y^2 + (3r^2 + 2ar + b)y + (r^3 + ar^2 + br + c) = 0$$

By equating any of the coefficients (other than the constant term) to zero, one can easily solve for r using an equation less than a cubic. For the y^2 term, we have $r = -a/3$. The method can obviously be applied to any *nth* degree equation.

For the cubic case, this was taken further by Francois Viete (1540-1603) when he managed to find a solution to the general cubic which required the extraction of only one cube root, a fact which was the inspiration for this paper.

An important contribution to extending the previous results was made in a short paper written in 1683 by Count Ehrenfried Walter von Tschirnhaus (1651-1708) who observed that this substitution can be generalized to higher degrees. Given the general equation,

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

the Cardano-Viete substitution was simply $y = x + b_0$. By allowing more general substitutions,

$$y = x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$$

we can potentially eliminate more than one term of any nth degree equation similar to how Viete managed to do for the general cubic. We get the transformed equation,

$$y^n + c_{n-1}y^{n-1} + \ldots + c_1 y + c_0 = 0$$

where *m* coefficients c_i can be eliminated, as the *m* parameters b_i enable us to fulfill *m* conditions. If we set *m* = *n-1* and all $c_i = 0$ (other than c_0) and by solving for the b_i, we can reduce the original equation to the binomial form,

$$y^n + c_0 = 0$$

Thus,

$$\sqrt[n]{-c_0} = x^{n-1} + b_{n-1}x^{n-2} + \ldots + b_1 x + b_0$$

an equation one degree less than the original. Tschirnhaus may have thought that by this gradual reduction, one degree at a time, all equations of any degree can be solved in radicals.

We can illustrate the procedure with the cubic. Given the depressed cubic,

$$x^3 + bx + c = 0$$

we can eliminate the x^1 term as well by using the quadratic Tschirnhausen transformation $y = x^2 + mx + n$ to get the cubic,

$$(y-(x_1{}^2+mx_1+n))(y-(x_2{}^2+mx_2+n))(y-(x_3{}^2+mx_3+n))=0$$

Collecting the new variable *y*, we have,

$$\begin{aligned}&y^3+(2b-3n)y^2+\left(b^2+3cm+bm^2-4bn+3n^2\right)y\\&+\left(-c^2+bcm+cm^3-b^2n-3cmn-bm^2n+2bn^2-n^3\right)=0\end{aligned}$$

Solving for the unknowns *m*, *n* to eliminate y^2 and y, we get,

$$\begin{aligned}m&=\frac{-9c\pm\sqrt{3(4b^3+27c^2)}}{6b}\\n&=\frac{2b}{3}\end{aligned}$$

and, using the negative case, we have the binomial cubic,

$$y^3-\frac{(4b^3+27c^2)\left(4b^3+27c^2+3c\sqrt{3(4b^3+27c^2)}\right)}{54b^3}=0$$

and since,

$$y=x^2+mx+n$$

by getting the cube root of the constant term, we have reduced the problem to solving a quadratic. It should be pointed out that the prevalence of the expression $4b^3+27c^2$ is understandable considering it is the *discriminant* of the cubic.

As applied to the general quintic to remove its y^4 and y^3 terms, this results in what is called the *principal quintic*. While the general quintic is not solvable in radicals, the principal quintic is solvable in the general case, as first done by Felix Klein (1849-1925), though one has to solve a related icosahedral equation and go beyond radicals and use hypergeometric functions.

The square root used to obtain the principal quintic is called by Klein the *accessory irrationality*, as it does not diminish the Galois group of the equation and as such, *is not expressible in terms of the roots of the equation*. This point will be very important to us later.

However, there was a problem with the Tschirnhausen transformation noticed by the philosopher-mathematician Gottfried Leibniz (1646-1716). The system of equations in the unknowns b_i, having degrees from *1* to *n-1*, was very hard to solve. From Bezout's theorem, (Etienne Bezout, 1730-1783) which states that the degree of the final equation of *m* complete equations in *m* unknowns is equal to the product of the degrees, then our final equation is of *(n-1)!* degree. Thus, for n > 3 we end up trying to solve an equation of much higher degree than the original.

There was a clever transformation though that allowed *three* terms to be eliminated from general equations of degree 5 and higher. This transformation is due to Erland Bring (1736-1798) with his work on quintics and independently, George Jerrard (1804-1863), who generalized it to higher degrees.

Later we will show that *three* terms can be eliminated from the general quartic as well, in contrast to what William Rowan Hamilton (1805-1865) thought as was mentioned in the paper "*Inquiry Into The Validity Of A Method Recently Proposed By George B. Jerrard, Esq., For Transforming And Resolving Equations of Elevated Degrees*".

To eliminate 3 terms from the general equation, the logical step was to use a third-degree Tschirnhausen transformation, namely,

$$y = x^3 + b_2 x^2 + b_1 x + b_0$$

We know that this will give us a system of 3 equations of degrees 1, 2, 3 and from Bezout's theorem, our final equation is of degree 3! = 6. However, Bring, and later, Jerrard, found a way around this obstacle. Instead of a cubic Tschirnhausen transformation, they used a quartic one of form,

$$y = x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0$$

The extra parameter allowed them to reduce the system of equations to eventually just solving a quadratic, then a cubic. This can then reduce the general quintic to the form,

$$y^5 + d_1 y + d_0 = 0$$

known as the Bring-Jerrard quintic. This form is also solvable in the general case though one has to use again hypergeometric functions. For a contemporary treatment of the details, the interested reader is referred to the paper by V. Adamchik and D. Jeffrey, "*Polynomial Transformations of Tschirnhaus, Bring, and Jerrard*".

To reduce the quintic beyond the Bring-Jerrard form, to the binomial form, as was mentioned in the abstract we would need an *n-1* Tschirnhausen transformation, hence also a quartic one. However, unlike the Bring-Jerrard form, since we have to eliminate *four* terms, we need all four b_i and do not have the luxury of an extra parameter to help in any simplification.

Our system then will be four equations in four unknowns, of degrees 1, 2, 3, 4, hence the final equation is of degree 4! = 24. Will this be solvable in radicals? In general, obviously not. But since we limited ourselves to *solvable equations*, we can prove, that for this subset of equations, this 24th degree equation is indeed also solvable.

III. Theorems

It has already been pointed out that one advantage modern mathematicians have over mathematicians of centuries past is the technological one: the access to computers, computer algebra systems, and the Internet. What past mathematicians didn't do due to the sheer amount of effort involved, we can let the computer do for us.

So to find whether a solvable equation, say, a quintic can be reduced to binomial form, we can actually try to resolve the system of 4 equations in the 4 unknowns b_i. For every solvable quintic tested by the author, the 24th degree was also solvable, i.e. it was reducible such that it had a factor less than the fifth degree.

However, what was desired was a general proof. How to prove that the 24th degree equation, for solvable quintics, had solvable factors? The author wrestled with the problem for a while until it was realized that there was another question easier to answer: *Can one prove that the roots of the 24th degree equation of the* b_i *were expressible in terms of the roots* x_i *of the solvable quintic*?

The answer is *yes*.

Theorem 1. Any irreducible but *solvable* equation Q(x) can be transformed into the binomial form $y^n + c_0 = 0$ in radicals using a Tschirnhausen transformation of degree *n-1*.

Proof:

What was realized was that since we had *n-1* equations in *n-1* unknowns, instead of focusing on the resultant (*n-1)*!-degree equation, what we were dealing with *was simply a matrix*.

Let us use the particular case of the quintic. We have:

$$Q(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

where it is desired to transform it into the form,

$$y^5 + c_0 = 0$$

using the quartic Tschirnhausen transformation,

$$y = x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$$

or specifically,

$$
\begin{aligned}
y_1 &= x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0 \\
y_2 &= x_2{}^4 + b_3 x_2{}^3 + b_2 x_2{}^2 + b_1 x_2 + b_0 \\
y_3 &= x_3{}^4 + b_3 x_3{}^3 + b_2 x_3{}^2 + b_1 x_3 + b_0 \\
y_4 &= x_4{}^4 + b_3 x_4{}^3 + b_2 x_4{}^2 + b_1 x_4 + b_0 \\
y_5 &= x_5{}^4 + b_3 x_5{}^3 + b_2 x_5{}^2 + b_1 x_5 + b_0
\end{aligned}
$$

It seems we have 5 equations in 4 unknowns. However, since $y^5 + c_0 = 0$, then we can express our y_i *in terms of the fifth root of* c_0 *and the fifth roots of unity.*

$$
\begin{aligned}
\sqrt[5]{-c_0} &= b_4 x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0 \\
\sqrt[5]{-c_0}\, w &= b_4 x_2{}^4 + b_3 x_2{}^3 + b_2 x_2{}^2 + b_1 x_2 + b_0 \\
\sqrt[5]{-c_0}\, w^2 &= b_4 x_3{}^4 + b_3 x_3{}^3 + b_2 x_3{}^2 + b_1 x_3 + b_0 \\
\sqrt[5]{-c_0}\, w^3 &= b_4 x_4{}^4 + b_3 x_4{}^3 + b_2 x_4{}^2 + b_1 x_4 + b_0 \\
\sqrt[5]{-c_0}\, w^4 &= b_4 x_5{}^4 + b_3 x_5{}^3 + b_2 x_5{}^2 + b_1 x_5 + b_0
\end{aligned}
$$

where w is any *complex* fifth root of unity.

By eliminating $\sqrt[5]{-c_0}$,

$$
\begin{aligned}
(x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0) w &= x_2{}^4 + b_3 x_2{}^3 + b_2 x_2{}^2 + b_1 x_2 + b_0 \\
(x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0) w^2 &= x_3{}^4 + b_3 x_3{}^3 + b_2 x_3{}^2 + b_1 x_3 + b_0 \\
(x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0) w^3 &= x_4{}^4 + b_3 x_4{}^3 + b_2 x_4{}^2 + b_1 x_4 + b_0 \\
(x_1{}^4 + b_3 x_1{}^3 + b_2 x_1{}^2 + b_1 x_1 + b_0) w^4 &= x_5{}^4 + b_3 x_5{}^3 + b_2 x_5{}^2 + b_1 x_5 + b_0
\end{aligned}
$$

then we really just have 4 *linear* equations in the 4 unknowns b_i! By using Gaussian elimination, the b_i can be solved for and expressed in terms of the x_i which are the roots of the solvable quintic. Therefore, the b_i are also roots of solvable equations, equations of the 24^{th} degree.

While we have used the particular case of the quintic, one can easily see that results can be applied to any degree and thus we have proven the theorem that a solvable equation of *any* degree can be transformed to binomial form in radicals.

The constant term to transform the quintic into the binomial form has a particularly nice form. Solving the system of equations above, we have,

$$
b_0 = \frac{z_1 x_2 x_3 x_4 x_5 - x_1 z_2 x_3 x_4 x_5 w + x_1 x_2 z_3 x_4 x_5 w^2 - x_1 x_2 x_3 z_4 x_5 w^3 + x_1 x_2 x_3 x_4 z_5 w^4}{z_1 - z_2 w + z_3 w^2 - z_4 w^3 + z_5 w^4}
$$

where,

$$
\begin{aligned}
z_1 &= (x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)(x_3 - x_5)(x_4 - x_5) \\
z_2 &= (x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_3 - x_4)(x_3 - x_5)(x_4 - x_5) \\
z_3 &= (x_1 - x_2)(x_1 - x_4)(x_1 - x_5)(x_2 - x_4)(x_2 - x_5)(x_4 - x_5) \\
z_4 &= (x_1 - x_2)(x_1 - x_3)(x_1 - x_5)(x_2 - x_3)(x_2 - x_5)(x_3 - x_5) \\
z_5 &= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)
\end{aligned}
$$

Before we go to the second theorem, we recall a statement made earlier, that Hamilton believed the general quartic could not be reduced to binomial form. (The same W. Hamilton who, when he first came up with the laws for the quaternions, on impulse etched it on the Brougham bridge.)

We quote from page 4 of the paper cited earlier, to wit, "*…the processes proposed by Mr. Jerrard … although valid as general transformations of the equation of the mth degree, become in general illusory when they are applied to resolve equations of the fourth and fifth degrees, by reducing them to the binomial form…*"

One interpretation of the statement is that the *particular* method by which Jerrard could eliminate 3 terms from the general quintic was believed by Hamilton as not applicable to the general quartic, although *other* methods may apply to eliminate 3 terms from the quartic and reduce it to binomial form.

For our method, which is just a straightforward Tschirnhausen transformation, our b_i would be roots of a 3! = 6-th degree equation with rational coefficients and generally irreducible. However, we have seen that the b_i would be expressible in terms of the roots x_i of the quartic, and hence, this sextic should be solvable. Thus, the general quartic indeed can be reduced to binomial form.

Theorem 2. Any solvable equation Q(x), with no repeated roots, can be transformed into *any* solvable form P(y) of the same degree in radicals using a Tschirnhausen transformation of degree n-1.

Proof:

The second theorem is the generalization of the first, since now we wish to turn the solvable *nth* degree equation not just into the binomial form, but into *any* solvable form of the same degree. While the Tschirnhausen transformation was developed in the context of eliminating intermediate terms in an equation by setting some of the $c_i = 0$, we can also set the c_i as equal to specific constants, thus transforming one equation into a desired equation.

However, since we are dealing with *n* coefficients c_i (excluding the leading coefficient which is assumed to be equal to one), then we have *n* equations in *n* unknowns and our final equation has *n*! degree.

Given,

$$Q(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

It is desired to transform it into the form,

$$P(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_1 y + c_0 = 0$$

with *n* coefficients c_i using the *n-1* degree Tschirnhausen relation,

$$y = b_n x^{n-1} + b_{n-1}x^{n-2} + \ldots + b_1 x + b_0$$

with *n* coefficients b_i.

For the particular case of the cubic we have,

$$y_1 = b_2 x_1^2 + b_1 x_1 + b_0$$
$$y_2 = b_2 x_2^2 + b_1 x_2 + b_0$$
$$y_3 = b_2 x_3^2 + b_1 x_3 + b_0$$

It is even more straightforward to see that the second theorem is just *n linear* equations in the *n* unknowns b_i which by Gaussian elimination can be expressed in terms of the x_i and y_i.

Solving for the b_i, we have,

$$b_2 = \frac{(x_2 - x_3)y_1 + (-x_1 + x_3)y_2 + (x_1 - x_2)y_3}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$
$$b_1 = \frac{(-x_2^2 + x_3^2)y_1 + (x_1^2 - x_3^2)y_2 + (-x_1^2 + x_2^2)y_3}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$
$$b_0 = \frac{(x_2^2 x_3 - x_2 x_3^2)y_1 + (-x_1^2 x_3 + x_1 x_3^2)y_2 + (x_1^2 x_2 - x_1 x_2^2)y_3}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

As one can see, the structure of the denominators justifies the prohibition of repeated roots of Q(x). The b_i are roots of 3! = 6-th degree equations with rational coefficients but since the x_i and y_i can be expressed in radicals, then so can the b_i.

Again, while we have used a particular case, this time the cubic, one can easily see that the basic idea can apply to *any* degree. While the b_i are roots of n! degree equations, *if the* x_i *and* y_i *can be expressed in radicals*, then this *n*! degree equation is also solvable and we have proven the second theorem.

There is an implication to the theorem that is already well-established, although it is usually not mentioned in the context of a Tschirnhausen transformation. Since we have proven that any solvable equation Q(x), with no repeated roots, can be transformed into any solvable form P(y) of the same degree in radicals, *then we can also transform the cyclotomic equation* $x^n = 1$ *into any solvable form*.

Consider the Lagrange resolvents of the roots y_i of some equation P(y),

$$u(w) = y_1 + y_2 w + y_3 w^2 + \ldots + y_n w^{n-1}$$

where w is any complex *nth* root of unity. For the specific case of the quintic, we have

$$u(w) = y_1 + y_2 w + y_3 w^2 + y_4 w^3 + y_5 w^4$$

A variation of the above is given by,

$$y_{k+1} = u_4 w^{4k} + u_3 w^{3k} + u_2 w^{2k} + u_1 w^k$$

for k = {0,1,2,3,4}. Or expressed another way,

$$y_{k+1} = u_4 (w^k)^4 + u_3 (w^k)^3 + u_2 (w^k)^2 + u_1 (w^k)$$

Or,

$$y_i = u_4 x^4 + u_3 x^3 + u_2 x^2 + u_1 x$$

where the x_i are the five roots of $x^5 = 1$. It is starting to look like a Tschirnhausen transformation. In fact, it *is* a Tschirnhausen transformation.

It is a theorem that if a quintic with rational coefficients is solvable, then its roots y_i are expressible in the form above, where the u_i are the 5^{th} roots of the roots of a quartic (the Lagrange resolvent) also with rational coefficients.

Thus, the solution of a solvable quintic P(y) in terms of its Lagrange resolvents is *nothing more than a Tschirnhausen transformation that transforms the cyclotomic equation* $x^5 = 1$ *into P(y).*

In general then, the solution of a solvable equation P(y) of ***prime*** degree *n* in terms of its Lagrange resolvents is just a Tschirnhausen transformation of degree *n-1* that transforms the cyclotomic equation $x^n = 1$ into P(y).

III. Examples

We can give some examples in transforming a solvable equation to binomial form, particularly in the quintic case. While we can derive the b_i by solving its matrix, as we did for b_0, it seems that the expressions would be horribly complex. We may desire a more aesthetic form and one way to do so is to derive the 24th degree equation and see if it has small factors. It seems Nature cooperates and we can in fact find relatively simple expressions for the b_i.

We will give two examples that illustrate different "structures" of the solvable quintic based on their quartic Lagrange resolvents: for the first, the resolvent is reducible while for the second, it is irreducible. For a simple way to find these Lagrange resolvents, the reader is referred to the paper *"An Easy Way To Solve The Solvable Quintic Using Two Sextics"* by the same author.

Example 1.

Given the solvable quintic,

$$3x^5 + 15x^3 + 60x^2 + 15x + 5 = 0$$

with discriminant = $\dfrac{5^5 2999^2}{3^4}$

Resolvent has a zero root and is given by:

$$(z+3)(z+9)(3z-1)=0$$

So,

$$x = -3^{1/5} - 3^{2/5} + (1/3)^{1/5} = -1.99483$$

We can also use our alternative method of reducing it to binomial form. We use the quartic Tschirnhausen transformation, with the unknown coefficients *m, n, p, q* as b_i,

$$y = x^4 + mx^3 + nx^2 + px + q$$

Expanding and collecting the new variable *y*,

$$\prod_{i=1}^{5}(y-(x_i^{\;4}+mx_i^{\;3}+nx_i^{\;2}+px_i+q))=0$$

where the x_i are the five roots of our quintic, we get,

$$y^5+c_4y^4+c_3y^3+c_2y^2+c_1y+c_0=0$$

where the c_i are polynomials in the unknowns *m, n, p, q*. By solving the system of equations where the c_i are set $c_4=c_3=c_2=c_1=0$, we would expect a final equation of the 24th degree.

In practice, the final equation gets a more elevated degree. Solving for *m* by eliminating one unknown at a time by getting their resultant, we have the sequence of degrees,

$$\{1,2,3,4\}\rightarrow\{2,3,4\}\rightarrow\{6,8\}\rightarrow\{48\}$$

thus, we end up with a 48th degree equation in *m* with rational coefficients. However it factors with a spurious 24th degree factor and the correct 24th degree one, which for this particular example has 4 linear factors, namely,

$$(-64+79m)(11+127m)(111+191m)(47+270m)=0$$

Using the first factor m = 64/79, we find the rest of the coefficients of the Tschirnhausen transformation,

$$y=x^4+\frac{64}{79}x^3+\frac{295}{79}x^2+\frac{5419}{237}x+\frac{884}{79}$$

and our binomial quintic is given by,

$$y^5+\left(\frac{1}{3}\right)\left(\frac{2999}{237}\right)^5=0$$

Since the discriminant *D* of a binomial quintic $y^5+f=0$ is simply $D=f^4$, then it shouldn't be surprising that the discriminant of the original quintic, $D=\frac{5^5 2999^2}{3^4}$, appears in the constant term.

Getting its fifth root, and equating it to the Tschirnahausen relation above, we then have,

$$-\frac{2999}{\sqrt[5]{3}}w=237x^4+192x^3+885x^2+5419x+2652$$

where w is any of the five 5^{th} roots of unity. Each of the five possible quartics will have one root in common with the original quintic.

However, since our 24^{th} degree equation has *four* linear roots, then there are *four* ways to define the coefficients of the Tschirnhausen transformation, the other three given by,

$$\frac{2999}{\sqrt[5]{3^2}}w = 381x^4 - 33x^3 + 2144x^2 + 7269x + 1606$$

$$-\frac{2999}{\sqrt[5]{3^3}}w = 270x^4 - 47x^3 + 1236x^2 + 5718x + 288$$

$$\frac{2999}{\sqrt[5]{3^4}}w = 191x^4 - 111x^3 + 941x^2 + 2912x - 596$$

Example 2.

Given the solvable quintic,

$$x^5 - x^3 + 2x^2 - 2x + 1 = 0$$

with discriminant = 47^2

Irreducible resolvent is given by:

$$z^4 + \frac{13}{5^2}z^3 + \frac{856}{5^6}z^2 + \frac{988}{5^{10}}z - \frac{1}{5^{15}} = 0$$

So,

$$z_1^{1/5} + z_2^{1/5} + z_3^{1/5} + z_4^{1/5} = -1.734691\ldots$$

We can do the same procedure as in the first example. However, this time, the correct 24^{th} degree equation has an irreducible quartic factor, namely,

$$19m^4 + 127m^3 - 369m^2 + 293m - 71 = 0$$

with one root given by,

$$m = \frac{\left(-127 + 83\sqrt{5} + \sqrt{94(745 - 331\sqrt{5})}\right)}{76}$$

Finding the rest of the coefficients of the Tschirnhausen transformation,

$$y = x^4 + mx^3 + nx^2 + px + q$$

we have,

$$n = \frac{\left(-78 + 42\sqrt{5} + \sqrt{94(125 - 41\sqrt{5})}\right)}{38}$$

$$p = \frac{\left(110 - 11\sqrt{5} + \sqrt{47(145 - 62\sqrt{5})}\right)}{38}$$

$$q = \frac{\left(-605 + 165\sqrt{5} - \sqrt{470(745 - 331\sqrt{5})}\right)}{190}$$

and our binomial quintic is the rather intimidating,

$$y^5 - \frac{47^2 v}{10 * 19^5} = 0$$

where,

$$v = -235(-1408830 + 627131\sqrt{5}) + \sqrt{47(4642619986\,334005 - 1038110165\,7422222 / \sqrt{5})}$$

we can again get its fifth root and the five values substituted into the Tschirnhausen relation above should give us five quartics, each with one root in common with the original quintic.

However, there should also be *four* ways to define the coefficients of the Tschirnhausen relation, as in the first example. The other variables *n, p, q* obviously are roots of quartics, namely,

$$19n^4 + 156n^3 - 61n^2 - 444n - 241 = 0$$
$$19p^4 - 220p^3 + 760p^2 - 935p + 305 = 0$$
$$95q^4 + 1210q^3 + 3220q^2 + 3080q + 976 = 0$$

as well as the c_0, the constant in the binomial. Its quartic is cumbersome to write down though it has a rather curious additional role, as we shall see later.

Let $r = 19^5 c_0$,

$$5^5 r^4 - 563532 * 5^6 47^3 r^3 + 1745038 * 5^4 19^5 47^5 r^2 - 41 * 5^3 19^{10} 47^8 r - 19^{15} 47^{10} = 0$$

Since it would be tedious to identify which root of one variable goes with which root of another, the efficient way would be to express the roots of the other four quartics in terms of the roots of the first. In other words, we seek to find a Tschirnhausen transformation to transform the first quartic into any of the other four.

This is where our second theorem comes in. We know it can be done. But again, it seems that we will just end up with complicated expressions.

In general, if we transform a random solvable equation into another random one, perhaps that is the case. But for this particular case, the five quartics are somehow "related". For one thing, they have a common factor in their discriminants. So, one can try to apply the second theorem to see what happens.

We would need a cubic Tschirnhausen transformation with 4 unknowns, namely,

$$v_i = am^3 + bm^2 + cm + d$$

where v_i in turn will be transformed into *n, p, q*, and c_0,

Let $v_i = n$

Expanding and collecting terms,

$$\prod_{i=1}^{4}(n - (am_i^{\,3} + bm_i^{\,2} + cm_i + d)) = 0$$

where the m_i are the roots of our quartic in *m*, we get,

$$n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0 = 0$$

where the c_i are polynomials in the unknowns *a, b, c, d*. Instead of setting the c_i as equal to zero, we set it to the coefficients of the *n* equation, namely,

$$c_3 = 156/19$$
$$c_2 = -61/19$$
$$c_1 = -444/19$$
$$c_0 = -241/19$$

Solving for the unknown *a* in this system of 4 equations in 4 unknowns, we again expect a *solvable* 24th degree equation. (In reality, it is also a 48th degree equation with a spurious 24th degree factor.)

Fortunately, Nature again cooperates with a sense of aesthetics of her own. The correct 24^{th} degree equation is not only solvable but it factors, with 4 linear factors and a 20^{th} degree equation,

$$(-6041+541a)(-931+541a)(2221+541a)(4751+541a) = 0$$

As we need only one, the logical choice would be the smallest factor,

$$(-931+541a) = 0$$

Since we now have one unknown, we can find the other three unknown coefficients of the Tschirnhausen relation, which turns out to be given by,

$$n = \frac{931m^3 + 7211m^2 - 10020m + 2498}{541}$$

The same process can be applied in turn to the other quartics and in the end we will have a more aesthetic set of relations.

So, given the solvable quintic,

$$x^5 - x^3 + 2x^2 - 2x + 1 = 0$$

Let the Tschirnhausen transformation be,

$$y = x^4 + mx^3 + nx^2 + px + q$$

where,

$$n = \frac{931m^3 + 7211m^2 - 10020m + 2498}{541}$$
$$p = \frac{1254m^3 + 9503m^2 - 16113m + 6434}{541}$$
$$q = -\frac{2(931m^3 + 7211m^2 - 11643m + 5203)}{5(541)}$$

and,

$$19m^4 + 127m^3 - 369m^2 + 293m - 71 = 0$$

and we have the binomial quintic,

$$y^5 - \left(\frac{47}{95}\right)^2 \frac{149687m^3 + 3942342m^2 - 5280911m + 1622466}{541} = 0$$

in four ways, depending on the root chosen of the *m* quartic.

IV. Conjectures.

While we have accomplished what we have set out to do, namely to establish certain broad and definitive statements about solvable equations, we can go into the finer details relevant to our topic. To complement our two theorems, we can make some remarks of a more speculative nature.

Conjecture 1. Let *n* be *prime.* Then the final equation with degree (*n-1*)! of the system of equations of the b_i used to transform a solvable equation Q(x) into the binomial form $y^n + c_0 = 0$ has a factor, with rational coefficients, of degree *n-1*.

We have already mentioned the Lagrange resolvents. It is a theorem that these resolvents, or their nth powers, are roots of equations with rational coefficients of degree *n-1*. Furthermore, these coefficients are determined by equations of degree *(n-2)*! which have, for solvable equations, a linear factor. In short, Lagrange resolvent equations are factors of a *(n-1)*! degree equation, just like our final equation.

Since both the Lagrange resolvents and our b_i are expressed in terms of the roots of the original equation and the *nth* roots of unity, then perhaps we can expect similar behavior in their defining polynomials as well and we can make our conjecture.

Conjecture 2. Let *n* be *prime.* If a solvable equation Q(x) is transformed into the binomial form $y^n \pm c_0 = 0$, then c_0 is a root of a Lagrange resolvent that solves another solvable *n*th degree equation R(y).

The author noticed this when deriving the quartic of c_0 (in the variable *r*). As one can see, its constant term is the fifth power $19^{15}47^{10}$. Since we have already pointed out in a previous paper *("An Easy Way…")* that the quartic Lagrange resolvent of the solvable quintic always has a constant term that is the fifth power of a rational number, this fact was suspicious.

It turned out that this quartic was indeed the resolvent of another quintic, namely,

$$R(y) = y^5 - 13395\,y^3 - 629565\,y^2 - 2142577579\,y - 123031 * 5^2 47^3 = 0$$

such that,

$$y = r_1^{1/5} + r_2^{1/5} + r_3^{1/5} + r_4^{1/5} = 258.05458824\ldots$$

For the cubic transformed into the binomial cubic, this was generally true. To recall, given the reduced cubic,

$$x^3 + bx + c = 0$$

We transform it to the binomial,

$$y^3 - \frac{(4b^3 + 27c^2)\left(4b^3 + 27c^2 + 3c\sqrt{3(4b^3 + 27c^2)}\right)}{54b^3} = 0$$

The unsigned constant term of the above is a root of the quadratic,

$$z^2 - \frac{(4b^3 + 27c^2)^2}{27b^3} z + \frac{(4b^3 + 27c^2)^3}{9^3 b^3} = 0$$

with a constant term that is a third power, which is not surprising. It turns out this is the Lagrange resolvent equation for the cubic,

$$x^3 - \frac{(4b^3 + 27c^2)}{3b} x - \frac{(4b^3 + 27c^2)^2}{27b^3} = 0$$

such that,

$$x = z_1^{1/3} + z_2^{1/3}$$

While the conjecture is true for n = 3 (and it is trivial for the case n = 2), is the conjecture generally true for all prime *n*? It seems an interesting question to ask.

V. Conclusion

The original four-page paper which Tschirnhaus wrote, probably between doing chemical experiments on porcelain and writing to the philosopher Spinoza, was entitled, in translation, *"A Method For Removing All Intermediate Terms From A Given Equation"*.

The primary objection to the method, as was first pointed out by Leibniz, was that to eliminate more and more intermediate terms, the final equation turns out to be of much higher degree than the original and apparently much harder to solve. There doesn't seem to any mention of whether this final equation was expressible or not in terms of *a)* the roots of the original equation or *b)* the roots of unity, nor whether this was a relevant point to consider.

In hindsight, that is understandable. The importance of the roots of unity would have to wait until Abraham de Moivre (1667-1754) and other mathematicians of the 18th century such as Leonhard Euler (1707-1783). The formula,

$$(\cos a + i \sin a)^n = \cos(na) + i \sin(na)$$

was first given by Euler only in 1748, though it is usually referred to as de Moivre's formula.

And with the work of Joseph-Louis Lagrange (1736-1813), another piece of the puzzle fell in place, namely the role of the permutations of the roots, though it was only in the early 19th century with the arrival of Abel and Galois that the complete picture emerged.

Tschirnhaus' method, addressed at trying to solve any equation of any degree, was certainly not true in the general case. However, as we have seen, for the class of equations that were solvable, the method in fact was valid.

Tschirnhaus would have been pleased.

-End-

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