"Ramanujan and Fifth Power Identities"

by Titus Piezas III

Abstract: We give the explicit formula to find solutions to $a_1^4 + a_2^4 + a_3^4 = 2b_1^{2m}$ for all positive integer *m*, an equation discussed by Ramanujan, as well as a generalization to third and fifth powers:

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 2b_1^{3m}$$

$$a_1^5 + a_2^5 + a_3^5 + a_4^5 + a_5^5 + a_6^5 = 2b_1^{5m}$$

Other quintic identities will also be discussed, including a *sum-product* analogous to the ones previously found for third and fourth powers.

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I. Introduction

In one of his Notebooks (Berndt, Ramanujan, Vol.4, p.96), Ramanujan gave an interesting set of identities,

$$2(ab+ac+bc)^2 = a^4 + b^4 + c^4$$
$$2(ab+ac+bc)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4$$
$$2(ab+ac+bc)^6 = (a^2b+b^2c+c^2a)^4 + (ab^2+bc^2+ca^2)^4 + (3abc)^4$$

where $a+b+c = 0$ (as well as one for $k = 8$) and adding the cryptic remark, "*…and so on*", though he didn't give the rest. Note that the first is equivalent to,

$$2(a^2+ab+b^2)^2 = a^4 + b^4 + (a+b)^4$$

One can find more identities of this sort using a basic idea that is beautifully simple. This was the approach used by Kevin Ford[1] to generalize Ramanujan's identities to all even powers,

$$2(a^2+ab+b^2)^{2k} = a^4 + b^4 + (a+b)^4$$

for any positive integer k (p. 100). The idea is that given an equal sums of like powers identity with a sum that is a bivariate quadratic polynomial $P(a,b)$, we are looking for expressions a,b such that $P(a,b)$ becomes a perfect power k, a polynomial $Q(p,q)^k$ of the same shape,

$$P(a,b) = Q(p,q)^k$$

By a clever trick, one can solve this equation and express a,b in terms of arbitrary p,q. This can be done by factoring over the *discriminant* of the quadratic to get linear factors,

$$v_1 v_2 = x_1^k x_2^k$$

and by equating $v_1 = x_1^k$, $v_2 = x_2^k$, we then get two linear equations in the two unknowns a,b which can then be solved for by simple Gaussian elimination. Ramanujan's identity was for fourth powers but analogous ones can also be given for third and fifth powers, with the former based on the *multi-grade* equation,

$$(a+c)^r + (a-c)^r + (b+c)^r + (b-c)^r = 2(a+b)^r$$

for $r = 1,2,3$ if $c = \sqrt{(ab)}$. Obviously, this can be rationalized by setting $a = a^2$, $b = b^2$, to get,

$$(a^2+ab)^r + (a^2-ab)^r + (b^2+ab)^r + (b^2-ab)^r = 2(a^2+b^2)^r$$

For fourth and fifth powers, we have,

$$a^m + b^m + (a+b)^m = 2(2c)^m$$

$$(a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n = 2(3c)^n$$

for $m = 2,4$ and $n = 1,3,5$, both of which are true if $c = (1/2)\sqrt{(a^2+ab+b^2)}$. To recall, the algebraic form a^2+b^2 is intimately connected to the *Gaussian integers*. On the other hand, a^2+ab+b^2 is connected to the *Eisenstein integers*.

II. 3rd Powers

To find a generalization of the cubic multi-grade we equate the polynomials,

$$(a^2+b^2) = (p^2+q^2)^k$$

and factor over their discriminant, which (square-free) is the imaginary unit $i = \sqrt{-1}$,

$$(a-bi)\,(a+bi) = (p-qi)^k\,(p+qi)^k$$

Equating factors,

$$(a-bi) = (p-qi)^k , \qquad (a+bi) = (p+qi)^k$$

we get two equations and *a*,*b* can then be easily solved for in terms of *p*,*q*.

"**Theorem 1**. The multi-grade equation,

$$(a^2+ab)^r + (a^2-ab)^r + (b^2+ab)^r + (b^2-ab)^r = 2(p^2+q^2)^{kr}$$

for r = 1,2,3 is solvable for any positive integer k using the formulas for *a*,*b*,

$$a = (x_1^k + x_2^k)/2, \qquad b = i(x_1^k - x_2^k)/2,$$

where $x_1 = (p-qi)$ and $x_2 = (p+qi)$ for arbitrary *p, q*."

Example. Let k = 2, then,

$$a = (p^2-q^2), \qquad b = 2pq,$$

while for k = 3,

$$a = p(p^2-3q^2), \qquad b = q(3p^2-q^2),$$

and so on. The same formulas for *a,b* can also be used to generalize the familiar second degree identity,

$$(a^2-b^2)^2 + (2ab)^2 = (p^2+q^2)^{2k}$$

for any positive integer k.

III. 4th Powers

The same approach can be applied to fourth powers. Since,

$$a^m + b^m + (a+b)^m = 2(a^2+ab+b^2)^{m/2}$$

for m = 2,4 we wish to find unknown expressions *a*,*b* such that,

$$a^2+ab+b^2 = (p^2+pq+q^2)^k$$

This has discriminant $\sqrt{-3}$ and while we can factor over this, a more elegant approach is to factor over a complex cube root ω of unity (any root of $\omega^2+\omega+1 = 0$) yielding,

$$(a-b\omega)(a-b\omega^2) = (p-q\omega)^k (p-q\omega^2)^k$$

and giving the two equations,

$$(a-b\omega) = (p-q\omega)^k, \qquad (a-b\omega^2) = (p-q\omega^2)^k$$

where the *a,b* are then solved for.

"**Theorem 2.** The multi-grade equation,

$$a^m + b^m + (a+b)^m = 2(p^2+pq+q^2)^{km/2}$$

for m = 2,4 can be solved for all positive integer k using the formulas for *a,b*,

$$a(-1+\omega) = \omega(p-q\omega)^k - (p-q\omega^2)^k, \qquad b(-1+\omega)\omega = (p-q\omega)^k - (p-q\omega^2)^k,$$

where ω is a complex cube root of unity and for arbitrary *p, q*."

Example. Let $k = 3$,

$$a = p^3-3pq^2-q^3, \qquad b = 3pq(p+q), \qquad a+b = p^3+3p^2q-q^3$$

and for $k = 4$,

$$a = p(p^3-6pq^2-4q^3), \quad b = q(4p^3+6p^2q-q^3), \qquad a+b = (p^2-q^2)(p^2+4pq+q^2)$$

and so on for all higher k. (It should be pointed out these are not identical to the more elegant expressions found by Ramanujan so he may have used a different method.)

IV. 5th Powers

Surprisingly, there is a fifth power multi-grade identity similar to the one for fourth powers and is given by,

$$(a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n = 2(3c)^n$$

for n =1,3,5 with $c = (1/2)\sqrt{(a^2+ab+b^2)}$. Since this also involves the form a^2+ab+b^2, then we can use our formulas for *a,b* derived in the previous section, though only the one for even $k = 2h$.

"**Theorem 3.** The multi-grade equation,

$$((a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n)(2/3)^n = 2(p^2+pq+q^2)^{hn}$$

for n = 1,3,5 can be solved for all positive integer h using the formulas for *a,b,c*,

$$a(-1+\omega) = \omega(p-q\omega)^{2h} - (p-q\omega^2)^{2h}, \qquad b(-1+\omega)\omega = (p-q\omega)^{2h} - (p-q\omega^2)^{2h},$$

$$c = (1/2)(p^2+pq+q^2)^h$$

where ω is a complex cube root of unity and for arbitrary *p, q*."

Notice that the expression originally $(3/2)^n$ was *factored to the left hand side,* as the right side now has the composite exponent *hn*. Appropriate choice of *p,q* can yield integral values for the addends. The reason for the limitation to even *k* is that *c* as given in the Introduction involves a square root $c = (1/2)\sqrt{(a^2+ab+b^2)}$. If we wish for it to be rational, since we already solved the equation,

$$a^2+ab+b^2 = (p^2+pq+q^2)^k$$

by letting $k = 2h$, and taking the square root of both sides, naturally *c* becomes,

$$c = (1/2)(p^2+pq+q^2)^h$$

Example. Let $h = 2$ and we find,

$$a = p(p^3-6pq^2-4q^3), \quad b = q(4p^3+6p^2q-q^3), \quad c = (1/2)(p^2+pq+q^2)^2$$

For $h = 3$, we get,

$$a = (p^3+3p^2q-q^3)(p^3-3p^2q-6pq^2-q^3)$$
$$b = 3pq(2p+q)(p+2q)(p^2-q^2)$$
$$c = (1/2)(p^2+pq+q^2)^3$$

and so on for all integral *h*.

V. Other Identities of the 5th Degree: Quintic Octuples

Let the equal sums of like powers,

$$a_1^k + a_2^k + \dots a_m^k = b_1^k + b_2^k + \dots b_n^k$$

be denoted as *k.m.n.* In the special case of *k* *k*th powers equal to a *k*th power, parametrizations are known for the odd powers $k = 3$ and 5, with the former found by Euler and the latter by Sastry (1934). There are also for 5.3.3 though typically they involve polynomials of high degree. If we limit ourselves to *binary quadratic forms,* these can solve the case 5.4.4 also known as *quintic octuples.* (In general, we will consider any $m+n = 8$ as an octuple.)

In a previous paper *"Ramanujan and the Quartic Equation $2^4+2^4+3^4+4^4+4^4 = 5^4$"* it was stated that an identity found in 1958 by Xeroudakes and Moessner (Lander, p.1069),

$$(p^2-4pq-9q^2)^k + (3p^2+16pq+17q^2)^k + (-p^2+13q^2)^k + (3p^2+8pq+q^2)^k + (-p^2-8pq-3q^2)^k + (p^2+12pq+23q^2)^k = 2(3(p^2+4pq+7q^2))^k$$

for k =1,3,5 was related to the ones for fourth powers found by Ramanujan. When the author first encountered this it was not arranged in this manner, but presented this way, its relationship to the other formulas becomes clear. This is just one case of the quintic multi-grade identity given earlier with,

$$a = 2(p^2+6pq+5q^2), \quad b = -8(pq+2q^2), \quad c = (1/2)\sqrt{(a^2+ab+b^2)}.$$

In fact, it can be proven that quintic octuples of form,

$$P(n) = (a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n + r^n + s^n = 0$$

for n = 1,3,5 have a complete parametrization. In other words, *a,b,c* must always satisfy a certain condition. By eliminating *r,s*, from the three equations $P(1)$, $P(3)$, $P(5)$ using resultants, one gets a final equation,

$$(a^2+ab+b^2-28c^2)(a^2+ab+b^2-4c^2) = 0$$

Thus, there are in fact *two* possible identities. Sparing the reader the rest of the algebra, these are given by,

$$(a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n = 2(3c)^n$$

where $a^2+ab+b^2 = 4c^2$, and,

$$(a+c)^n + (b+c)^n + (a+b+c)^n + (-a-b+c)^n + (-b+c)^n + (-a+c)^n + c^n = (7c)^n$$

where $a^2+ab+b^2 = 28c^2$, both for n = 1,3,5. (Interestingly, the latter algebraic form appears again in an identity for *sixth* powers, though we are getting ahead of ourselves.) These two were nearly found by Kawada and Wooley when they gave the algebraic identity

$$(h+x)^5 + (h-x)^5 + (h+y)^5 + (h-y)^5 + (h+x+y)^5 + (h-x-y)^5 = 20h(x^2+xy+y^2+h^2)^2 - 14h^5$$

for arbitrary *h,x,y*. Another kind of *k*.4.4 quadratic form identity[2], multi-grade for k =1,2,3,5 can be given by,

$$p_1^k + p_2^k + p_3^k + p_4^k = p_5^k + p_6^k + p_7^k + p_8^k$$

where,

$$p_1 = (-a+b+c)x^2 + 2(cu-bv)xy - (a+b+c)uvy^2$$
$$p_2 = (a-b+c)x^2 + 2(cu+bv)xy + (a+b-c)uvy^2$$
$$p_3 = (a+b-c)x^2 + 2(-cu-bv)xy + (a-b+c)uvy^2$$
$$p_4 = -(a+b+c)x^2 + 2(-cu+bv)xy + (-a+b+c)uvy^2$$
$$p_5 = -(a+b+c)x^2 + 2(-bu+av)xy + (a+b-c)uvy^2$$
$$p_6 = (a-b+c)x^2 + 2(-bu-av)xy + (-a+b+c)uvy^2$$
$$p_7 = (-a+b+c)x^2 + 2(bu+av)xy + (a-b+c)uvy^2$$
$$p_8 = (a+b-c)x^2 + 2(bu-av)xy - (a+b+c)uvy^2$$

and $u = a^2-b^2$, $v = b^2-c^2$, for five free variables, *a,b,c,x,y*. A side condition this identity obeys is,

$$p_1 + p_4 = -(p_2 + p_3), \qquad p_5 + p_8 = -(p_6 + p_7),$$

and a similar octuple was studied by Lander (p. 1067) which depended on solving,

$$a_1a_2a_3(a_1^2 + a_2^2 + a_3^2) = b_1b_2b_3\,(b_1^2 + b_2^2 + b_3^2)$$

Finally, a multi-grade for k = 1,2,3,4,5, can be given by the two k.6.6,

$$(a_1x+v_1y)^k + (a_2x-v_2y)^k + (a_3x+v_3y)^k + (a_4x-v_3y)^k + (a_5x+v_2y)^k + (a_6x-v_1y)^k =$$
$$(a_1x-v_1y)^k + (a_2x+v_2y)^k + (a_3x-v_3y)^k + (a_4x+v_3y)^k + (a_5x-v_2y)^k + (a_6x+v_1y)^k$$

$$(a_1x^2+2v_1xy+3a_6y^2)^k + (a_2x^2-2v_2xy+3a_5y^2)^k + (a_3x^2+2v_3xy+3a_4y^2)^k + (a_4x^2-2v_3xy+3a_3y^2)^k$$
$$+ (a_5x^2+2v_2xy+3a_2y^2)^k + (a_6x^2-2v_1xy+3a_1y^2)^k = (a_1^k+ a_2^k+a_3^k+a_4^k+a_5^k+a_6^k)(x^2+3y^2)^k$$

where $\{a_1, a_2, a_3, a_4, a_5, a_6\} = \{a+c, b+c, -a-b+c, a+b+c, -b+c, -a+c\}$, and $\{v_1, v_2, v_3\} = \{a+2b, 2a+b, a-b\}$ for five arbitrary variables *a,b,c,x,y*.

The last one is a *sum-product* and is analogous to the ones for third and fourth powers and were derived in the same manner, namely by expanding the equation, collecting powers of *x,y*, and solving for the v_i, though certain heuristics based on numerical examples were also used. Note that the v_i are the very same ones for fourth powers. The usefulness of such a sum-product is that one can easily find parametrizations for *any* $m+n > 6$ by just being given one solution to the equation,

$$a_1^5 + a_2^5 + a_3^5 + a_4^5 + a_5^5 + a_6^5 = z$$

with the a_i as defined above and where z can be decomposed as a sum and difference of a number of fifth powers. By replacing the first factor of the right hand side with z, then it guarantees that there will be an infinite number of solutions. For example, starting with the trivial,

$$(-1)^5 + (-1)^5 + 1^5 + 1^5 + 3^5 + 3^5 = 2(3)^5$$

using $\{a,b,c\} = \{-2,2,1\}$, we get the not-so-trivial,

$$(-x^2+4xy+9y^2)^k + (-x^2-4xy+9y^2)^k + (x^2+8xy+3y^2)^k + (x^2-8xy+3y^2)^k + (3x^2+4xy-3y^2)^k + (3x^2-4xy-3y^2)^k = 2(3x^2+9y^2)^k$$

for $k = 1,3,5$.

VI. Conclusion

We can end this work with some questions:

1. *What other identities of quintic octuples depend on binary quadratic forms?*
2. *Are there any other quintic sum-product identities?*

Two examples of octuples were given with terms p_i as linear functions in three variables a,b,c and where the three only had a quadratic relationship, depending either on $a^2+ab+b^2 = 4c^2$, or $a^2+ab+b^2 = 28c^2$. But surely there must be more. The two identities have an appealing Zen simplicity that makes one want to find more of the same kind. And considering the sum-product given was dependent on the basic form shared by the two, finding another one might imply another kind of sum-product.

For *sixth* powers there is a k.4.4 identity found by Chernick in 1937,

$$(a-7c)^k + (a-2b+c)^k + (3a+c)^k + (3a+2b+c)^k = (a+7c)^k + (a-2b-c)^k + (3a-c)^k + (3a+2b-c)^k$$

for $k = 2,4,6$ and where $a^2+ab+b^2 = 7c^2$. By letting $c = 2c$, this is same algebraic form employed in the second quintic identity! It might be interesting to know if there is a relationship between the two. Again, there must be other k.4.4 identities for 6^{th} powers dependent on binary quadratic forms but this seems to be the only one known so far.

Care to find a few?

--End--

Footnotes:

1. The author wishes to thank Kevin Ford who, in an email, pointed me in the right direction for this paper.
2. This identity is based on one I found in 2004 in a website "*Equal sums of four fifth powers*" at http://www.crossnumbers.co.za/four1.htm which was only for k = 1,3,5. It seems the site is no longer active.

May 4, 2006
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