“On A Connection Between Solvable Quintics And Fibonacci Numbers”

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ABSTRACT: We explore a hitherto unsuspected connection between certain solvable quintics with rational coefficients whose parametrization involves Fibonacci numbers as a particular case, but entails the solution of the Pell equations $x^2 - Dy^2 = \pm 4$ in the general case. An additional connection to Pythagorean triples will also be discussed.

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India, 605 AD

The astronomer-mathematician was sitting in front of his desk in his small library, trying to solve a mathematical problem that he had been working on all morning. While trying yet another approach, he noticed a small movement behind the open door in the doorway.

“Come in, little one,” he invited. “You have been there for some time already.”

“I don’t want to disturb you, sir,” was the soft reply.

“Not at all. Come inside.”

“What are you studying, sir?” he asked, as he approached the desk.

The old man smiled tolerantly as he sought the words to answer in an understandable manner. “I am looking for a number such that if I add one to a multiple of a square, it will result in another square. Here, take a look.”

The explanation would have daunted a lesser child, but the precocious little boy, not quite ten years old, was in his lessons already showing promise of being a great scholar one day. He looked at the scribbles which centuries later would be understood as the Pell equation $1+61y^2 = x^2$.

“Would there be an answer to it?” he asked.

“I have found solutions for smaller multiples than 61, but not for this one.”

“Maybe there is, but perhaps the numbers are very big.”

The old man was pleased with the answer. It was a true mathematician who believed that the present absence of evidence was not evidence of permanent absence.

“Yes, they could be very large numbers, indeed.”

There was a thoughtful silence then the boy spoke up. “Maybe one day I’ll find those numbers.”

The old man smiled encouragingly and patted him on the head. “If you apply yourself to your lessons, then maybe you will.”

At that moment, they heard somebody outside the room call out a name.

“Bramagupta? Where are you? You are not done with your chores yet,” a female voice stated.

The little boy winced, much preferring his studies than doing the household chores.

A woman appeared in the doorway of the library. “There you are! Now, don’t be disturbing your grandfather, Bramagupta,” she admonished. “Off you go now.”

The boy reluctantly followed his mother as they left the room.

Years later, when the boy became a man, he would solve that equation and many other mathematical problems, and would become a leading figure of the mathematics of the time. Indeed there was an answer, though it involved numbers far larger than one would have guessed.

Around a thousand years later, halfway across the globe, a mathematician in the French town of Toulouse would rediscover the very same equation, and extensively study the general case. This man, however, is more famous for solving another Diophantine mathematical problem, or so he claimed, the proof of which he famously didn't write in the margins of a book…

I. Introduction

It is already well established that general equations of degree greater than four are not solvable in radicals, i.e. they cannot be solved using a finite number of arithmetic operations and root extractions. However, there are particular equations of degree greater than four that can be solvable in radicals.

In *"An Easy Way To Solve The Solvable Quintic Using Two Sextics"* [1] by the same author, a simple procedure was proposed to find the explicit expressions for the roots of the quintic when it is solvable. In this paper, we will discuss two ways such that we can come up with infinite examples of solvable quintics on which we can apply the procedure. The first way involves Pell equations, and in a particular case the Fibonacci numbers, and the second involves Pythagorean triples. Both have been studied for centuries, in fact the latter one going back millennia to the ancient Babylonians, though certainly not in the context of quintic equations.

Pell equations have been studied by the Indian mathematicians Bramagupta (598-670 AD) and Bhaskara (1114-1185). In the West, the systematic study of them started with Pierre de Fermat (1601-1665) who fanned the flames of interest on them by issuing a challenge to solve certain Pell equations. It was Leonard Euler (1707-1783) who gave the equation its name, after the mathematician John Pell (1611-1685), though it seems the connection between Pell and his eponymous equation is rather tenuous.

Pell equations are primarily of the form $x^2 - Dy^2 = 1$ and secondarily, $x^2 - Dy^2 = \pm 4$, where x and y are to be solved in the integers. They are important in mathematics, appearing in various contexts from number theory (for one, solutions give excellent rational approximations to $\sqrt{D}$) and to class field theory in the calculation of fundamental units in an algebraic field.

Fibonacci numbers, named after Leonardo Fibonacci (1170-1250), are the sequence of numbers F = {1,1,2,3,5,8,13,21, etc} where, starting with 1, each succeeding number is the sum of the two numbers preceding it. A closed-form formula will be mentioned later.

These numbers have a lot of interesting properties. As first described in Fibonacci's book *Liber Abaci*, they give the number of pairs of rabbits n months after one pair begins to breed. However, these numbers appear in other mathematical contexts, such as in Pascal's Triangle, polynomial sequences, etc, not to mention its connection with the golden section, $f = \frac{1+\sqrt{5}}{2} = 1.61803...$, a ratio known to the ancient Greeks and supposedly with aesthetically pleasing qualities.

Pythagorean triples are integral solutions to the Diophantine equation $a^2 + b^2 = c^2$. This equation has even a longer history than Pell equations. While traditionally named after Pythagoras (569-475 BC), there is evidence that the ancient Babylonians of c. 2000 BC were aware of them.

This equation also appears in a trigonometric context as relating the hypotenuse of a right triangle with its other two sides. An extension of it, $a^n + b^n = c^n$, was thought by Fermat as not solvable in the integers for n > 2, the proof of which he claimed he had though didn't write down because the margin of the book, Diophantus' *Arithmetica*, was too narrow to contain it. Thus tantalizing generations of mathematicians until Andrew Wiles' proof in 1994.

II. The Solvable Quintic

In the paper cited earlier [1], we derived two sextics that factors when the quintic is irreducible but solvable. They were derived from two equations in two unknowns, the variables *p* and *t*, by eliminating one unknown to have a single equation in the other unknown.

We can cite those two equations again. For the details in how we arrived at those equations, the reader is referred to [1] where they are given as equations (15) and (16).

Let the reduced quintic be,

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0$$

Equation (15) is,

$$(t^2 - c^3 - d^2 + ce)ct + (dt + 10c^2p + ep - 25p^3)pt + (-d^3 + 2cde - c^2f + fp^2)p = 0 \qquad (1)$$

while (16) is,

$$\begin{aligned}&(t^2 - c^3 - d^2 + ce)^2 - p^2(11c^4 + 2cd^2 - 35c^2p^2 + 25p^4 - 16dpt + 14ct^2)\\ &- p^2(4c^2e - 6ep^2 + e^2) = 0\end{aligned} \qquad (2)$$

Eliminating *t* we have the equation in *p*,

$$\begin{aligned}&(3125p^6 - 625(3c^2 + e)p^4 + 25(15c^4 + 8cd^2 - 2c^2e + 3e^2 - 2df)p^2 - 25c^6\\ &- 40c^3d^2 - 16d^4 + 35c^4e + 28cd^2e - 11c^2e^2 + e^3 - 2c^2df - 2def + cf^2)^2 - Dp^2 = 0\end{aligned} \qquad (3)$$

where D is the discriminant,

$$\begin{aligned}D = &-3200c^3d^2e^2 - 2160d^4e^2 + 6400c^4e^3 + 5760cd^2e^3 - 2560c^2e^4 + 256e^5\\ &+ 5120c^3d^3f + 3456d^5f - 11520c^4def - 10080cd^3ef + 4480c^2de^2f - 640de^3f\\ &+ 3456c^5f^2 + 2640c^2d^2f^2 - 1440c^3ef^2 + 360d^2ef^2 + 160ce^2f^2 - 120cdf^3 + f^4\end{aligned}$$

One can easily see that by expanding (3) and collecting powers of f, then it has degree 4 where the coefficient of f^4 is $c^2 - p^2$. By letting $p = c$, we can reduce (3) to a polynomial of degree 3.

$$\begin{aligned}
&4cd(4c^2 - e)f^3 - 2(128c^7 - 192c^4d^2 + 16cd^4 - 80c^5e + 48c^2d^2e + 16c^3e^2 - 2d^2e^2 \\
&- ce^3)f^2 - 4d(41600c^8 + 5440c^5d^2 + 448c^2d^4 - 17920c^6e - 1632c^3d^2e - 16d^4e \\
&+ 2144c^4e^2 + 28cd^2e^2 - 70c^2e^3 + e^4)f - 8e^3(232c^3d^2 + 4d^4 - 7cd^2e + 16c^2e^2) \\
&+ 128(20000c^{12} + 4000c^9d^2 - 200c^6d^4 - 40c^3d^6 + 2d^8 - 16000c^{10}e - 900c^7d^2e \\
&+ 230c^4d^4e - 7cd^6e + 4800c^8e^2 - 95c^5d^2e^2 + 7c^2d^4e^2 - 665c^6e^3 + 42c^4e^4) + e^6 = 0
\end{aligned} \tag{4}$$

Thus, one should find rational c, d, e such that (4) has a rational root f. While that may seem imposing, we still have (1) and (2). Since $p = c$, we have (1) as,

$$c(d^3 - 2cde + 16c^3t + d^2t - 2cet - dt^2 - t^3) = 0$$

and (2) as,

$$(d - t)(d^3 - 2cde + 16c^3t + d^2t - 2cet - dt^2 - t^3) = 0$$

where e is just linear and easily solved for. If we substitute this value of e into (4), amazingly the cubic factors with a linear root f!

Before doing so, we can re-define our variables to have a more aesthetically pleasing form. Let,

$$d = \frac{a+b}{2}$$

$$t = \frac{-a+b}{2}$$

$$c = \frac{r}{2}$$

and e becomes,

$$e = \frac{ab^2 - ar^3 + br^3}{br}$$

and (4) factors into,

$$\left(f(ab^2r^2) - a^3b^3 + (5a^2 - 5ab + b^2)b^2r^3 - a^2r^6\right)(Q) = 0 \tag{5}$$

where Q is a quadratic in f, a long expression rather tedious to write down. One can easily derive it by using a computer algebra system. Solving for the linear root, we have our solvable quintic in the variables a, b, r,

$$x^5 + 5rx^3 + 5(a+b)x^2 + 5\left(\frac{ab^2 - ar^3 + br^3}{br}\right)x + \frac{a^3b^3 - (5a^2 - 5ab + b^2)b^2r^3 + a^2r^6}{ab^2r^2} = 0$$

However, we can always ask: *Can we make (4) factor completely into linear rational factors*? This is where the connection to Pell equations, and as a natural consequence, to Fibonacci numbers, would come in.

The discriminant of the quadratic factor is given by:

$$D_4 = b(a^2b + 4ar^3 + 4br^3)$$

Thus, the objective is to make D_4 a perfect square h^2,

$$a^2b^2 + 4abr^3 + 4b^2r^3 = h^2$$

Or,

$$h^2 - (a^2 + 4r^3)b^2 = 4abr^3 \tag{6}$$

Let,

$$\begin{aligned} h &= axy \\ b &= ay^2 \end{aligned} \tag{7}$$

So,

$$(axy)^2 - (a^2 + 4r^3)(ay^2)^2 = 4a(ay^2)r^3$$

Eliminating common factors,

$$x^2 - (a^2 + 4r^3)y^2 = 4r^3$$

Let r =1,

$$x^2 - (a^2 + 4)y^2 = 4 \tag{8.1}$$

Let r = -1

$$x^2 - (a^2 - 4)y^2 = -4 \tag{8.2}$$

And we have our Pell equations! We can also define h, b a second way.

Let,

$$\begin{aligned} h &= xy \\ b &= y^2 \end{aligned} \tag{9}$$

So,

$$(xy)^2 - (a^2 + 4r^3)(y^2)^2 = 4a(y^2)r^3$$

Or,

$$x^2 - (a^2 + 4r^3)y^2 = 4ar^3$$

Let r =1,

$$x^2 - (a^2 + 4)y^2 = 4a \qquad (10.1)$$

Let r = -1

$$x^2 - (a^2 - 4)y^2 = -4a \qquad (10.2)$$

And we have two Pell-like equations.

III. Solutions To Pell Equations

We can mention two basic results about these equations. First, the form $x^2 - Dy^2 = 1$ has integral solutions for *any* non-square positive integer *D*. Second it has an *infinity* of solutions which can be derived from the fundamental solution.

Let *p, q* be the fundamental solutions. We then have,

$$x^2 - Dy^2 = (p^2 - Dq^2)^n = 1$$

Factoring both sides,

$$\left(x + y\sqrt{D}\right)\left(x - y\sqrt{D}\right) = \left(p + q\sqrt{D}\right)^n \left(p - q\sqrt{D}\right)^n$$

So,

$$x + y\sqrt{D} = \left(p + q\sqrt{D}\right)^n$$
$$x - y\sqrt{D} = \left(p - q\sqrt{D}\right)^n$$

And we have 2 equations in the 2 unknowns *x, y*. Solving for *x, y* gives the infinite number of solutions,

$$x = \frac{\left(p + q\sqrt{D}\right)^n + \left(p - q\sqrt{D}\right)^n}{2}$$

$$y = \frac{\left(p+q\sqrt{D}\right)^n - \left(p-q\sqrt{D}\right)^n}{2\sqrt{D}}$$

A related result is if the equation $x^2 - Dy^2 = k$ for some non-zero integral k has a solution, then it also has an infinite number of solutions.

Let (a, b) be the fundamental solutions to $x^2 - Dy^2 = k$ and (p, q) the solutions to $x^2 - Dy^2 = 1$. Then we have the identity:

$$\left(a^2 - Db^2\right)\left(p^2 - Dq^2\right) = \left(ap \pm Dbq\right)^2 - D\left(aq \pm bp\right)^2 = k$$

Since there are an infinite p, q, then we can generate infinite solutions such that $x^2 - Dy^2 = k$. However, since there may be several "fundamental" a, b, then there may be more than one infinite family.

As a side note, we can remark than that the above identity is nothing more than the Fibonacci Two-Square Identity in disguise, an identity which states that the product of two sums, both a sum of two squares, is also a sum of two squares.

This identity is very important as it is fundamental to much of trigonometry, as well as being the basis of a division algebra, along with Euler's Four-Square Identity and Degen-Grave's Eight-Square Identity (and no more). Consider,

$$\left(a^2 + m^2\right)\left(p^2 + n^2\right) = \left(ap - mn\right)^2 + \left(an + mp\right)^2$$

Let,

$$m = b\sqrt{-D}$$
$$n = q\sqrt{-D}$$

Then,

$$\left(a^2 - Db^2\right)\left(p^2 - Dq^2\right) = \left(ap + Dbq\right)^2 - D\left(aq + bp\right)^2$$

the same identity equal to k as above.

A. For $x^2 - (a^2 + 4)y^2 = 4$ (8.1)

The solution of (8.1) is given by:

$$x_0 = \frac{\left(a^2 + 2 + a\sqrt{a^2+4}\right)^n + \left(a^2 + 2 - a\sqrt{a^2+4}\right)^n}{2^n}$$

$$y_0 = \frac{\left(a^2 + 2 + a\sqrt{a^2+4}\right)^n - \left(a^2 + 2 - a\sqrt{a^2+4}\right)^n}{2^n\sqrt{a^2+4}}$$

We can provide the first few values of *y* for n = 1,2,3,4, etc.

$$a$$
$$2a + a^3$$
$$3a + 4a^3 + a^5$$
$$4a + 10a^3 + 6a^5 + a^7$$

Since from (7) we have $b = ay^2$, and by substituting $r = 1$ and any of our values of b into Q--the quadratic in the variable *f* in (5)--or equivalently into (4) taking into account the intermediate substitutions, then we can have three rational values for *f*.

Supposing we let $b = a(a)^2 = a^3$, then we have the triplet of solvable quintics:

$$x^5 + 5x^3 + 5(a + a^3)x^2 + 5\left(\frac{a^6 + a^2 - 1}{a^2}\right)x + f_i = 0$$

where,

$$f_1 = \frac{1 - 5a^6 + 5a^8}{a^5}$$

$$f_2 = \frac{-1 - 14a^2 - 20a^4 - 25a^6 - 13a^8 - 6a^{10}}{a^3(1 + a^2 + a^4)}$$

$$f_3 = \frac{-1 - 12a^2 - 3a^4 + 15a^6 + 3a^8 - 8a^{10} + a^{12} + a^{14}}{a^3(-1 + a^4)}$$

The solution to $x^2 - (a^2 + 4)y^2 = 4$ is where the connection, or at least half of the connection, to Fibonacci numbers comes in. By setting $a = 1$ we have,

$$x^2 - 5y^2 = 4$$

where,

$$x_0 = \frac{\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n}{2^n}$$
$$y_0 = \frac{\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

first few values of y are {1,3,8,21,55, etc}, which one should recognize as a bisection of the Fibonacci numbers F = {1,1,2,3,5,8,13,21,34, etc}, or every other number. Since $b = ay^2$, where we have set $a = 1$, then we have an infinite number of triplet quintics solvable in radicals.

This bisection is featured in N.J. Sloane's Online Encyclopedia of Integer Sequences, ID No. A001906, where one can see other mathematical contexts in which this sequence appears, other than this new connection to solvable quintics.

Since this is half of all Fibonacci numbers, what about the other half? It will come up in the next Pell equation.

B. For $x^2 - (a^2 - 4)y^2 = -4$ (8.2)

It is in the solutions to $x^2 - (a^2 - 4)y^2 = -4$ that the other half of the connection of solvable quintics to Fibonacci numbers appears.

We cannot provide a parametric solution to (8.2) in terms of a as it seems there may be only a finite number of integral values such that it is solvable. For a < 100, the only value the author found was $a = 3$. So we have,

$$x^2 - 5y^2 = -4$$

whose solutions are given by,

$$x_0 = \frac{\left(1+\sqrt{5}\right)^n + \left(1-\sqrt{5}\right)^n}{2^n}$$
$$y_0 = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

only for **odd** n, as we are dealing with the constant –4.

First few values of y for odd n are {1,2,5,13,34, etc} and we have the other half of the bisection of the Fibonacci numbers, as was mentioned earlier.

In fact, our y_0 is the closed-form formula for the Fibonacci numbers, using both odd and even n, and is also known as *Binet's Fibonacci Number formula* after the French mathematician Jacques Philippe Binet (1786-1856).

For example, let $y = 2$. Since $a = 3$, and from (7) we have $b = ay^2$, then $b = 12$. Taking into account we are using the negative case of r this time, $r = -1$, then we have the triplet of solvable quintics,

$$x^5 - 5x^3 + 75x^2 - \frac{705}{4}x + f_i = 0$$

where,

$$f_1 = \frac{3545}{8}, f_2 = \frac{5329}{48}, f3 = \frac{54089}{232}$$

C. For $x^2 - (a^2 + 4)y^2 = 4a$ (10.1)

We can come up with a parametric solution to (10.1). A fundamental solution to (10.1) is,

$$x_{01} = a + 2$$
$$y_{01} = 1$$

though it must be pointed out that for certain values of *a* the above may not be the only fundamental solution. Since the solutions to (8.1) are given by x_0, y_0, let

$$\frac{x_0}{2} = x_1, \qquad \frac{y_0}{2} = y_1$$

$$(2x_1)^2 - (a^2 + 4)(2y_1)^2 = 4$$

or,

$$x_1^2 - (a^2 + 4)y_1^2 = 1$$

So we can use the solutions of (8.1) to generate an infinite number of solutions to (10.1) using a version of the Fibonacci Two-Square Identity, as was discussed earlier.

Values for y_i are given by $\frac{x_{01}y_0 \pm y_{01}x_0}{2}$. Taking the negative case, for n = 0,1,2,3, etc,

$$-1$$
$$-1 + a$$
$$-1 + 2a - a^2 + a^3$$
$$-1 + 3a - 3a^2 + 4a^3 - a^4 + a^5$$

Taking the positive case,

$$1$$
$$1 + a + a^2$$
$$1 + 2a + 3a^2 + a^3 + a^4$$
$$1 + 3a + 6a^2 + 4a^3 + 5a^4 + a^5 + a^6$$

Since from (9) we have $b = y^2$, and by substituting any of our values for *b* into (5), then we can have another three rational values for f.

Supposing we let $b = (-1+a)^2$, and for simplicity a further substitution $a = v+1$, then we have the triplet of solvable quintics:

$$x^5 + 5x^3 + 5(1+v+v^2)x^2 + 5\left(\frac{-1-v+v^2+v^4+v^5}{v^2}\right)x + f_i = 0$$

where,

$$f_1 = \frac{1+2v+v^2-5v^4-10v^5+v^6+8v^7+2v^8+v^9}{v^4(1+v)}$$

$$f_2 = \frac{-1-3v-17v^2-25v^3-31v^4-35v^5-22v^6-12v^7-5v^8+v^9+v^{10}}{v^3(1+v)(1+v^2)}$$

$$f_3 = \frac{-1-v-12v^2-4v^3-4v^4+14v^5+2v^6+2v^7-8v^8+v^9+v^{10}+v^{11}}{v^3(-1+v)(1+v+v^2)}$$

D. For $x^2 - (a^2-4)y^2 = -4a$ (10.2)

We cannot provide a parametric solution to (10.2) in terms of integral a. For a < 100, the only value the author found was $a = 5$. So we have,

$$x^2 - 21y^2 = -20$$

an equation which has 3 fundamental solutions x_i, y_i, namely,

$$x_1 = 1, y_1 = 1$$
$$x_2 = 8, y_2 = 2$$
$$x_3 = 13, y_3 = 3$$

Let,

$$x^2 - 21y^2 = 1$$

with solutions,

$$x_0 = \frac{\left(55+12\sqrt{21}\right)^n + \left(55-12\sqrt{21}\right)^n}{2}$$
$$y_0 = \frac{\left(55+12\sqrt{21}\right)^n - \left(55-12\sqrt{21}\right)^n}{2\sqrt{21}}$$

So the solutions of $x^2 - 21y^2 = -20$ are given by,

$$x = x_i x_0 \pm 21 y_i y_0$$
$$y = x_i y_0 \pm y_i x_0$$

where x_i, y_i are any of the 3 fundamental solutions for any n.

First few values of y are {1,2,3,9,14,43,etc}. Let $y = 2$. Since from (9) we have $b = y^2$, then $b = 4$, and since $a = 5$, $r = -1$, then we have the triplet of solvable quintics:

$$x^5 - 5x^3 + 45x^2 - \frac{405}{4}x + f_i = 0$$

where,

$$f_1 = \frac{6901}{72}, f_2 = \frac{8681}{80}, f_3 = \frac{14981}{136}$$

IV. Pythagorean Triples

The second way involves the Pythagorean triples. The basic result we can mention is that the complete parametrization for $a^2 + b^2 = c^2$ is known. In fact, we can use the Fibonacci Two-Square Identity again.

$$(a^2 + m^2)(p^2 + n^2) = (ap - mn)^2 + (an + mp)^2$$

Let,

p = a
n = m

Then,

$$(a^2 + m^2)^2 = (a^2 - m^2)^2 + (2am)^2$$

The connection to Pythagorean triples is more apparent. That is, if one knows a certain theorem. In [1] we establish the third theorem in the paper that the pair of quintics with 4 free parameters *m, n, v, w*,

$$y^5 + 5(m+n)y^3 + 5(v+w)y^2 + 5e_1 y + f_1 = 0$$
$$y^5 + 5(m+n)y^3 + 5(v+w)y^2 + 5e_2 y + f_2 = 0$$

where,

$$e_{1\&2} = m^2 - mn + n^2 + \frac{(m+n)vw \pm (m-n)R_1R_2}{2mn}$$

and,

$$f_{1\&2} = 5(v-w)(m-n) + \frac{(2n^3v + 2m^3w)mn + (m^2v + n^2w)vw \pm (m^2v - n^2w)R_1R_2}{2m^2n^2}$$

and,

$$R_1R_2 = \sqrt{(4m^2n + v^2)(4mn^2 + w^2)}$$

are solvable for all *m, n, v, w* where $mn \neq 0$.

To find a solvable quintic with rational coefficients, the objective then is to find rational R_1R_2, though it should be pointed out that that is not the only way. For certain quadratic irrationals *m, n, v, w*, various examples of which were given in [1], the quintic can still end up with rational coefficients.

However, if we limit ourselves to rational *m, n, v, w*, one rather obvious option is to set $m = n = 1$. Then we have,

$$R_1R_2 = \sqrt{(4+v^2)(4+w^2)}$$

And the problem reduces to the sum of two squares equal to a square. The parametrization for that is easily given, simply a variation of the one mentioned above, though strictly speaking we are now shifting to rational, and not integral, solutions.

Let,

$$v = \frac{a_1^2 - b_1^2}{a_1b_1}$$
$$w = \frac{a_2^2 - b_2^2}{a_2b_2}$$

where a_1, a_2, b_1, b_2 are rational numbers. Then we have the rational number,

$$R_1R_2 = \left(\frac{a_1^2 + b_1^2}{a_1b_1}\right)\left(\frac{a_2^2 + b_2^2}{a_2b_2}\right)$$

And the theorem is modified such that the pair of quintics,

$$y^5 + 10y^3 + 5(v+w)y^2 + 5(vw+1)y + f_i = 0$$

where,

$$f_1 = \frac{(v+w)(vw+2) + (v-w)R_1R_2}{2}$$
$$f_2 = \frac{(v+w)(vw+2) - (v-w)R_1R_2}{2}$$

and,

$$v = \frac{a_1^{\,2} - b_1^{\,2}}{a_1 b_1}$$

$$w = \frac{a_2^{\,2} - b_2^{\,2}}{a_2 b_2}$$

$$R_1 R_2 = \left(\frac{a_1^{\,2} + b_1^{\,2}}{a_1 b_1}\right)\left(\frac{a_2^{\,2} + b_2^{\,2}}{a_2 b_2}\right)$$

are solvable for all non-zero a_1, a_2, b_1, b_2 and thus the connection between solvable quintics and Pythagorean triples is established.

V. Summary

One of the fascinations of the mathematical world is that it seems to be such a seamless tapestry, with connections between widely separated topics. What do numbers that originally arose in the context of breeding rabbits, or those that measure the sides of a triangle, have to do with solving equations using only a finite number of arithmetic operations and root extractions? Yet the connection is there, one only has to make the effort to look.

It is reminiscent of the paintings of the Dutch artist M. C. Escher (1898-1972), an artist known for his works of illusion such that one doesn't know whether stairs are ascending or descending, or for his symmetry paintings where images dissolve either from one kind or color into another. One painting comes into mind, Liberation, where from a gray sheet of paper gradually emerges connected white and black birds which eventually fly separately. For a more visual experience, one is referred to http://www.etropolis.com/escher/scroll.htm

Perhaps mathematical fields are like that. They appear separate, they look black and white, but in the end, all mathematical laws must be written in the same sheet of paper. It is hoped that this present work of the author has provided even just a small glimpse of one of these connections.

Because after all, there is also a connection between Fibonacci numbers and certain elliptic curves, but that is another story…

Author's Note:

Certain liberties have been taken with the details of Bramagupta's life. For one thing, there is virtually no evidence that his grandfather was still living when he was a young boy, much less that his grandfather introduced Pell equations to him.

However, it is not outside the realm of possibility that Bramagupta built on the mathematical work of older family members. The educational system of India at this time was primarily family-based, with families of astrologer-mathematicians passing their knowledge from one generation to the next. He would eventually become prominent in Ujjain, which was a leading mathematical center of India for 700 years, from c. 500-1200 AD.

Bramagupta did solve $x^2 - 61y^2 = 1$, as well as Bhaskara after him, and Fermat, who used it as a challenge to spur interest in the topic. Why the interest in this particular Pell equation? If one takes a look at a table of fundamental solutions for D<100, this has the largest fundamental solution with y = 226,153,980. The next largest is for D = 97, with y a measly 35 times less, roughly. It was no mean achievement. Try expressing that number in Roman numerals, which was the system used in Europe at this time.

As a last remark, we can mention the name of the man who introduced to Europe what are known as the Hindu-Arabic numerals. And who was this man?

Why, the same man whose name appears in the title. Leonardo Fibonacci.

-End-

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